Coupling the Supersymmetric 210 Vector Multiplet to Matter in SO(10)

Pran Nath and Raza M. Syed

Department of Physics, Northeastern University, Boston, MA 02115-5000, USA

Abstract

An analysis of the couplings of the 210 dimensional SO(10) vector multiplet to matter is given. Specifically we give an $SU(5) \times U(1)$ decomposition of the vector couplings $\overline{16}_{\pm} - 16_{\pm} - 210$, where $16_{\pm}$ is the semispinor of $SO(10)$ chirality $\pm$, using a recently derived basic theorem. The analysis is carried out using the Wess-Zumino gauge. However, we also consider the more general situation where all components of the vector multiplet enter in the couplings with the chiral fields. Here elimination of the auxiliary fields leads to a sigma model type nonlinear Lagrangian. Interactions of the type analysed here may find applications in effective theories with the 210 vector arising as a condensate. The analysis presented here completes the explicit computation of all lowest order couplings involving the $16_{\pm}$ of spinors with Higgs and vectors multiplets using the basic theorem.

1   Introduction

In the usual couplings of vector bosons, the vector bosons belong to the adjoint representation of the gauge group ($\widehat{V}_a$, a=1,..,N) and thus have one to one correspondence with the number of generators of the gauge group (T^a, a=1,..,N). This allows one to form the Lie valued quantity $\widehat{V} = \widehat{V}_a T^a$ which enters prominently in the construction of Yang-Mills gauge interactions which describe the self interactions of the gauge bosons. Further, one also utilizes the Lie valued quantities to couple the vector bosons to matter. In supersymmetric theories one essentially uses the same strategy in that one also uses Lie valued quantities and further one uses the Wess-Zumino gauge[1] in which the vector multiplet is reduced to just three components, $\mathcal{V}^\mu, \lambda, D$, where $\mathcal{V}^\mu$ is the spin 1 vector field, λ is a spin $\frac{1}{2}$ Majorana fields and D is an auxiliary field. The question arises how one may construct the couplings of a vector multiplet which does not belong to the adjoint representation. We focus here on the group SO(10) which is one of the groups under considerable scrutiny as it is a possible grand unification group for the unification of the electroweak and of the strong interactions[2]. Thus, for example, in SO(10) one has a 16 dimensional spinor representation which can accomodate a full one generation of quarks and leptons and its vector couplings have the following decomposition

$$\overline{16} \times 16 = 1 + 45 + 210 \tag{1}$$

Thus while it is straightforward to couple 1 and 45 plet of vectors with the $\overline{16} \times 16$ using the usual Yang-Mills construction, and for the case of supersymmetry using the supersymmetric Yang-Mills construction, the same procedure does not apply to the coupling of the vector 210 multiplet. Recently, we have given a complete computation of the couplings in the superpotential which involve the 16 plet of matter[3, 4]. In Ref.[3, 4] a "basic theorem" using oscillator method[5, 6] was developed which allowed one to carry out explicit analytic computations of the SO(10) couplings. Since $16 \times 16 = 10 + 120 + \overline{126}$ we have given a complete determination of the couplings of matter -matter -Higgs couplings of the type $16 - 16 - 10$, $16 - 16 - 120$ and $16 - 16 - \overline{126}$ [7, 8]. The present analysis is motivated by similar considerations where we wish to give a complete analysis of the vector couplings of $\overline{16} \times 16$. While the vector couplings $\overline{16} - 16 - 1$ and $\overline{16} - 16 - 45$ are straightforwardly given by the standard analysis, this is not the case for the $\overline{16} - 16 - 210$ vector coupling. Here we need a new technique to address this question. The purpose of this paper is to do just that. In this paper

we consider the couplings of the supersymmetric vector 210 multiplet in SO(10). We focus on this construction both for the theoretical challenge of constructing such couplings as well as for the possibility that such interactions may surface in some future effective theories to describe fully all the degrees of freedom at some relevant energy scale.

The outline of the rest of the paper is as follows: In Sec.2 we follow the conventional approach and give the coupling of the 210 multiplet with 16 plet of matter, i.e., we compute the couplings $\overline{16}_{\pm} - 16_{\pm} - 210$ in the Wess-Zumino gauge and we carry out a full $SU(5) \times U(1)$ decomposition of it. Elimination of the auxiliary fields is carried out in Appendix C. At the very outset we discard the constraint of gauge invariance since the imposition of such a constraint is untenable for the 210 multiplet. In Sec.3 we consider the more general couplings of the 210 multiplet retaining all the components of the vector multiplet, i.e, we do not impose the Wess-Zumino gauge constraint[1]. In the construction we use the superfield formalism[9] to guarantee that we have explicit supersymmetry at all stages in the theory. In this case elimination of the auxiliary fields leads to a non-linear Lagrangian with infinite order of nonlinearities in it. The general technique underlying this procedure is illustrated in Appendix G for the U(1) case. This analysis has some resemblance to the analysis of Ref.[10] which also used a unconstrained vector multiplet, i.e., it did not impose the constraint of the Wess-Zumino gauge[1]. However, the analysis of Ref.[10] did not include an explicit mass term for the vector multiplet, nor the self interactions of the vector fields and it did not integrate the auxiliarly fields. In fact the motivation of the work of Ref.[10] was very different in that the analysis of Ref.[10] was geared to study spontaneous symmetry breaking and generation of vector boson masses in that context. Returning to the 210 multiplet we note that since the 210 vector multiplet interaction cannot be gauge invariant, one must view its interactions only as effective interactions and thus the appearance of sigma model type nonlinearities here are quite acceptable. In Sec.4 we give the conclusions. Appendix A is devoted to notation and definition of the components of the vector and chiral superfields. Normalization of the dynamical modes are given in Appendix B. An elimination of the auxiliary fields appearing in Sec.2 is given Appendix C and an illustration of the SO(10) couplings of Sec.3 is given in Appendix D. For completeness an $SU(5) \times U(1)$ decomposition of the singlet vector couplings and of the 45 vector couplings are given in Appendices E and F.

2 Coupling of 210 vector multiplet to $16_\pm$ plet of matter

In usual formulations of particle interactions the vectors belong to either singlets or the adjoint representations of the gauge group of the theory under consideration. To couple the vectors to matter one forms a Lie valued quantity $\widehat{V}^a T_a$ where T_a are the generators of the gauge group satisfying the algebra $[T_a, T_b] = i f_{abc} T_c$. Then one couples the Lie valued quantity to the matter fields in the form $\Phi^\dagger e^{g\widehat{V}} \Phi$ which can be shown to be a gauge invariant combination. For the representations 1 and 45 on the right hand side of Eq.(1) one can carry out this construction straightforwardly (see Appendices E and F). However, this construction does not work for the 210 vector multiplet as one cannot write a gauge invariant Yang-Mills theory for it. Further, for the same reason one cannot write a gauge invariant coupling of the 210 vector with matter. To construct the 210 vector couplings, the technique we adopt is to carry out a direct expansion in powers of the vector supersuperfield. Thus we have

$$
\begin{aligned}
\mathsf{L}_{V+\Phi}^{(210\ Interaction)} &= h_{ab}^{(210+)}[< \widehat{\Phi}_{(+)a}|\widehat{\Phi}_{(+)b} > + \frac{\mathsf{g}^{(210)}}{4!} < \widehat{\Phi}_{(+)a}|\widehat{V}_{\mu\nu\rho\lambda}\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|\widehat{\Phi}_{(+)b} > \\
&\quad + \frac{1}{2}\left(\frac{\mathsf{g}^{(210)}}{4!}\right)^2 < \widehat{\Phi}_{(+)a}|\widehat{V}_{\mu\nu\rho\lambda}\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}\widehat{V}_{\alpha\beta\gamma\delta}\Gamma_{[\alpha}\Gamma_\beta\Gamma_\gamma\Gamma_{\delta]}|\widehat{\Phi}_{(+)b} >]|_{\theta^2\bar\theta^2} \\
&\quad + h_{ab}^{(210-)}[< \widehat{\Phi}_{(-)a}|\widehat{\Phi}_{(-)b} > + \frac{\mathsf{g}^{(210)}}{4!} < \widehat{\Phi}_{(-)a}|\widehat{V}_{\mu\nu\rho\lambda}\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|\widehat{\Phi}_{(-)b} > \\
&\quad + \frac{1}{2}\left(\frac{\mathsf{g}^{(210)}}{4!}\right)^2 < \widehat{\Phi}_{(-)a}|\widehat{V}_{\mu\nu\rho\lambda}\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}\widehat{V}_{\alpha\beta\gamma\delta}\Gamma_{[\alpha}\Gamma_\beta\Gamma_\gamma\Gamma_{\delta]}|\widehat{\Phi}_{(-)b} >]|_{\theta^2\bar\theta^2} + .. \\
&\qquad\qquad \Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]} = \frac{1}{4!}\sum_P (-1)^{\delta_P}\Gamma_{\mu_{P(1)}}\Gamma_{\nu_{P(2)}}\Gamma_{\rho_{P(3)}}\Gamma_{\lambda_{P(4)}} \quad (2)
\end{aligned}
$$

where $\sum_P$ denoting the sum over all permutations and δ_P takes on the value 0 (1) for even (odd) permutations. $\widehat{V}_{\mu\nu\rho\lambda}$ (μ, ν, λ, ρ=1,2,..,10) is the vector superfield, $\widehat{\Phi}_{(+)a}$ is the 16_+ chiral superfield (a is the generation index) and Γ_μ satisfies a rank-10 Clifford algebra, $[\Gamma_\mu, \Gamma_\nu] = 2\delta_{\mu\nu}$. In the following we give a full exhibition of the couplings to only linear order in the vector superfield in terms of its $SU(5) \times U(1)$ decompostion but a similar analysis can be done for couplings involving higher powers of the superfield. Thus using the analysis of Ref.[3, 4] we find that the $\overline{16}_+ 16_+$ couplings can be decomposed as follows

$$h_{ab}^{(210+)} < \widehat{\Phi}_{(+)a}|\widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2} = h_{ab}^{(210+)}[-\partial_A\mathbf{A}_{(+)a}^\dagger\partial^A\mathbf{A}_{(+)b} - \partial_A\mathbf{A}_{(+)a}^{i\dagger}\partial^A\mathbf{A}_{(+)bi}$$
$$-\partial_A\mathbf{A}_{(+)aij}^\dagger\partial^A\mathbf{A}_{(+)b}^{ij} - i\overline{\mathbf{\Psi}}_{(+)aL}\gamma^A\partial_A\mathbf{\Psi}_{(+)bL}$$
$$-i\overline{\mathbf{\Psi}}_{(+)aL}^i\gamma^A\partial_A\mathbf{\Psi}_{(+)biL} - i\overline{\mathbf{\Psi}}_{(+)aijL}\gamma^A\partial_A\mathbf{\Psi}_{(+)bL}^{ij}] + \mathsf{L}_{(1)auxiliary}^{(210)}$$
$$\mathsf{L}_{(1)auxiliary}^{(210)} = h_{ab}^{(210+)} < F_{(+)a}|F_{(+)b} > \quad (3)$$

$$h_{ab}^{(210-)} < \widehat{\Phi}_{(-)a}|\widehat{\Phi}_{(-)b} > |_{\theta^2\bar{\theta}^2} = h_{ab}^{(210-)}[-\partial_A\mathbf{A}_{(-)a}^\dagger\partial^A\mathbf{A}_{(-)b} - \partial_A\mathbf{A}_{(-)ai}^\dagger\partial^A\mathbf{A}_{(-)b}^i$$
$$-\partial_A\mathbf{A}_{(-)a}^{ij\dagger}\partial^A\mathbf{A}_{(-)bij} - i\overline{\mathbf{\Psi}}_{(-)aL}\gamma^A\partial_A\mathbf{\Psi}_{(-)bL}$$
$$-i\overline{\mathbf{\Psi}}_{(-)aiL}\gamma^A\partial_A\mathbf{\Psi}_{(-)bL}^i - i\overline{\mathbf{\Psi}}_{(-)aL}^{ij}\gamma^A\partial_A\mathbf{\Psi}_{(-)bijL}] + \mathsf{L}_{(2)auxiliary}^{(210)}$$
$$\mathsf{L}_{(2)auxiliary}^{(210)} = h_{ab}^{(210-)} < F_{(-)a}|F_{(-)b} > \quad (4)$$

$$\mathbf{\Psi}_{(\pm)a} = \begin{pmatrix} \psi_{(\pm)a\tilde{\alpha}} \\ \overline{\psi}_{(\pm)a}^{\dot{\tilde{\alpha}}} \end{pmatrix} \quad (5)$$

In the above the upper case Latin letters (A, B, C, D) are the Lorentz indices while the Greek letters with tilde's($\tilde{\alpha}, \dot{\tilde{\beta}}, ...$)are Weyl indices. Similarly we find

$$\frac{h_{ab}^{(210+)}\mathsf{g}^{(210)}}{4!} < \widehat{\Phi}_{(+)a}|\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|\widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2}$$

$$= h_{ab}^{(210+)}\mathsf{g}^{(210)}\{[\frac{1}{2}\sqrt{\frac{5}{6}}\left(i\mathbf{A}_{(+)a}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b} - \overline{\mathbf{\Psi}}_{(+)aL}\gamma_A\mathbf{\Psi}_{(+)bL}\right)$$

$$-\frac{1}{4\sqrt{30}}\left(i\mathbf{A}_{(+)aij}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{ij} - \overline{\mathbf{\Psi}}_{(+)aijL}\gamma_A\mathbf{\Psi}_{(+)bL}^{ij}\right)$$

$$+\frac{1}{2\sqrt{30}}\left(i\mathbf{A}_{(+)a}^{i\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)bi} - \overline{\mathbf{\Psi}}_{(+)aL}^i\gamma_A\mathbf{\Psi}_{(+)biL}\right)]\mathcal{V}'^A$$

$$+[\frac{1}{\sqrt{6}}\left(i\mathbf{A}_{(+)a}^{i\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b} - \overline{\mathbf{\Psi}}_{(+)aL}^i\gamma_A\mathbf{\Psi}_{(+)bL}\right)]\mathcal{V}_i'^A$$

$$+[\frac{1}{\sqrt{6}}\left(i\mathbf{A}_{(+)a}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)bi} - \overline{\mathbf{\Psi}}_{(+)aL}\gamma_A\mathbf{\Psi}_{(+)biL}\right)]\mathcal{V}'^{Ai}$$

$$+[-\frac{1}{4\sqrt{2}}\left(i\mathbf{A}_{(+)alm}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b} - \overline{\mathbf{\Psi}}_{(+)almL}\gamma_A\mathbf{\Psi}_{(+)bL}\right)$$

$$+\frac{1}{24\sqrt{2}}\epsilon_{ijklm}\left(i\mathbf{A}_{(+)a}^{i\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{jk} - \overline{\mathbf{\Psi}}_{(+)aL}^i\gamma_A\mathbf{\Psi}_{(+)bL}^{jk}\right)]\mathcal{V}'^{Alm}$$

$$+[-\frac{1}{4\sqrt{2}}\left(i\mathbf{A}_{(+)a}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{lm} - \overline{\mathbf{\Psi}}_{(+)aL}\gamma_A\mathbf{\Psi}_{(+)bL}^{lm}\right)$$

$$+\frac{1}{24\sqrt{2}}\epsilon^{ijklm}\left(i\mathbf{A}_{(+)aij}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)bk} - \overline{\mathbf{\Psi}}_{(+)aijL}\gamma_A\mathbf{\Psi}_{(+)bkL}\right)]\mathcal{V}_{lm}'^A$$

$$+[\frac{1}{6\sqrt{2}}\left(i\mathbf{A}_{(+)aik}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{kj} - \overline{\mathbf{\Psi}}_{(+)aikL}\gamma_A\mathbf{\Psi}_{(+)bL}^{kj}\right)$$

$$-\frac{1}{2\sqrt{2}}\left(\mathbf{A}^{j\dagger}_{(+)a}\overleftrightarrow{\partial}_A\mathbf{A}_{(+)bi}+i\overline{\mathbf{\Psi}}^{j}_{(+)aL}\gamma_A\mathbf{\Psi}_{(+)biL}\right)]\mathcal{V}^{'iA}_{j}$$

$$+[\frac{1}{6\sqrt{6}}\epsilon_{ijklm}\left(i\mathbf{A}^{i\dagger}_{(+)a}\overleftrightarrow{\partial}_A\mathbf{A}^{jn}_{(+)b}-\overline{\mathbf{\Psi}}^{i}_{(+)aL}\gamma_A\mathbf{\Psi}^{jn}_{(+)bL}\right)]\mathcal{V}^{'Aklm}_{n}$$

$$+[-\frac{1}{6\sqrt{6}}\epsilon^{ijklm}\left(i\mathbf{A}^{\dagger}_{(+)ain}\overleftrightarrow{\partial}_A\mathbf{A}_{(+)bj}-\overline{\mathbf{\Psi}}_{(+)ainL}\gamma_A\mathbf{\Psi}_{(+)bjL}\right)]\mathcal{V}^{'An}_{klm}$$

$$+[-\frac{1}{4\sqrt{6}}\left(i\mathbf{A}^{\dagger}_{(+)aij}\overleftrightarrow{\partial}_A\mathbf{A}^{kl}_{(+)b}-\overline{\mathbf{\Psi}}_{(+)aijL}\gamma_A\mathbf{\Psi}^{kl}_{(+)bL}\right)]\mathcal{V}^{'Aij}_{kl}$$

$$-\frac{i}{2}\sqrt{\frac{5}{6}}\left[-\mathbf{A}^{\dagger}_{(+)a}\overline{\mathbf{\Psi}}_{(+)bR}+\frac{1}{10}\mathbf{A}^{\dagger}_{(+)aij}\overline{\mathbf{\Psi}}^{ij}_{(+)bR}-\frac{1}{5}\mathbf{A}^{i\dagger}_{(+)a}\overline{\mathbf{\Psi}}_{(+)biR}\right]\Lambda^{'}_{L}$$

$$+\frac{i}{\sqrt{3}}\left[\mathbf{A}^{i\dagger}_{(+)a}\overline{\mathbf{\Psi}}_{(+)bR}\right]\Lambda^{'}_{iL}+\frac{i}{\sqrt{3}}\left[\mathbf{A}^{\dagger}_{(+)a}\overline{\mathbf{\Psi}}_{(+)biR}\right]\Lambda^{'i}_{L}$$

$$-\frac{i}{4}\left[\mathbf{A}^{\dagger}_{(+)alm}\overline{\mathbf{\Psi}}_{(+)bR}-\frac{1}{6}\epsilon_{ijklm}\mathbf{A}^{i\dagger}_{(+)a}\overline{\mathbf{\Psi}}^{jk}_{(+)bR}\right]\Lambda^{'lm}_{L}$$

$$-\frac{i}{4}\left[\mathbf{A}^{\dagger}_{(+)a}\overline{\mathbf{\Psi}}^{lm}_{(+)bR}-\frac{1}{6}\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)aij}\overline{\mathbf{\Psi}}_{(+)bkR}\right]\Lambda^{'}_{lmL}$$

$$-\frac{i}{2}\left[-\frac{1}{3}\mathbf{A}^{\dagger}_{(+)aik}\overline{\mathbf{\Psi}}^{kj}_{(+)bR}+\mathbf{A}^{j\dagger}_{(+)a}\overline{\mathbf{\Psi}}_{(+)biR}\right]\Lambda^{'i}_{jL}$$

$$+\frac{i}{6\sqrt{3}}\epsilon_{ijklm}\left[\mathbf{A}^{i\dagger}_{(+)a}\overline{\mathbf{\Psi}}^{jn}_{(+)bR}\right]\Lambda^{'klm}_{nL}-\frac{i}{6\sqrt{3}}\epsilon^{ijklm}\left[\mathbf{A}^{\dagger}_{(+)ain}\overline{\mathbf{\Psi}}_{(+)bjR}\right]\Lambda^{'n}_{klmL}$$

$$-\frac{i}{4\sqrt{3}}\left[\mathbf{A}^{\dagger}_{(+)aij}\overline{\mathbf{\Psi}}^{kl}_{(+)bR}\right]\Lambda^{'ij}_{klL}$$

$$-\frac{i}{2}\sqrt{\frac{5}{6}}\left[-\overline{\mathbf{\Psi}}_{(+)aL}\mathbf{A}_{(+)b}+\frac{1}{10}\overline{\mathbf{\Psi}}_{(+)aijL}\mathbf{A}^{ij}_{(+)b}-\frac{1}{5}\overline{\mathbf{\Psi}}^{i}_{(+)aL}\mathbf{A}_{(+)bi}\right]\Lambda^{'}_{R}$$

$$-\frac{i}{\sqrt{3}}\left[\overline{\mathbf{\Psi}}^{i}_{(+)aL}\mathbf{A}_{(+)b}\right]\Lambda^{'}_{iR}-\frac{i}{\sqrt{3}}\left[\overline{\mathbf{\Psi}}_{(+)aL}\mathbf{A}_{(+)bi}\right]\Lambda^{'i}_{R}$$

$$+\frac{i}{4}\left[\overline{\mathbf{\Psi}}_{(+)almL}\mathbf{A}_{(+)b}-\frac{1}{6}\epsilon_{ijklm}\overline{\mathbf{\Psi}}^{i}_{(+)aL}\mathbf{A}^{jk}_{(+)b}\right]\Lambda^{'lm}_{R}$$

$$+\frac{i}{4}\left[\overline{\mathbf{\Psi}}_{(+)aL}\mathbf{A}^{lm}_{(+)b}-\frac{1}{6}\epsilon^{ijklm}\overline{\mathbf{\Psi}}_{(+)aijL}\mathbf{A}_{(+)bk}\right]\Lambda^{'}_{lmR}$$

$$+\frac{i}{2}\left[-\frac{1}{3}\overline{\mathbf{\Psi}}_{(+)aikL}\mathbf{A}^{kj}_{(+)b}+\overline{\mathbf{\Psi}}^{j}_{(+)aL}\mathbf{A}_{(+)bi}\right]\Lambda^{'i}_{jR}\}$$

$$-\frac{i}{6\sqrt{3}}\epsilon_{ijklm}\left[\overline{\mathbf{\Psi}}^{i}_{(+)aL}\mathbf{A}^{jn}_{(+)b}\right]\Lambda^{'klm}_{nR}+\frac{i}{6\sqrt{3}}\epsilon^{ijklm}\left[\overline{\mathbf{\Psi}}_{(+)ainL}\mathbf{A}_{(+)bj}\right]\Lambda^{'n}_{klmR}$$

$$+\frac{i}{4\sqrt{3}}\left[\overline{\mathbf{\Psi}}_{(+)aijL}\mathbf{A}^{kl}_{(+)b}\right]\Lambda^{'ij}_{klR}\}+\mathsf{L}^{(210)}_{(3)auxiliary}$$

$$\mathsf{L}^{(210)}_{(3)auxiliary}=\frac{h^{(210+)}_{ab}\mathsf{g}^{(210)}}{4!2}<A_{(+)a}|\Gamma_{[\mu}\Gamma_{\nu}\Gamma_{\rho}\Gamma_{\lambda]}|A_{(+)b}>D_{\mu\nu\rho\lambda} \qquad (6)$$

where

$$\mathbf{A}^{i\dagger}_{(+)a}\overleftrightarrow{\partial}_A\mathbf{A}_{(+)bi}\overset{def}{=}\mathbf{A}^{i\dagger}_{(+)a}\partial_A\mathbf{A}_{(+)bi}-\left(\partial_A\mathbf{A}^{i\dagger}_{(+)a}\right)\mathbf{A}_{(+)bi}$$

5

$$\Lambda_{R,L} = \frac{1 \pm \gamma_5}{2}\Lambda$$

$$\Lambda^i_{jkl} = \begin{pmatrix} \lambda^i_{\tilde{\alpha}jkl} \\ \overline{\lambda}^{\dot{\alpha}i}_{jkl} \end{pmatrix} \tag{7}$$

and so on. Similarly for the $\overline{16}_- 16_-$ couplings we find

$$\frac{h^{(210-)}_{ab}\mathsf{g}^{(210)}}{4!} < \widehat{\Phi}_{(-)a}|\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|\widehat{\Phi}_{(-)b} > |_{\theta^2\bar{\theta}^2}$$

$$= h^{(210-)}_{ab}\mathsf{g}^{(210)}\{[\frac{1}{2}\sqrt{\frac{5}{6}}\left(i\mathbf{A}^\dagger_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)b} - \overline{\boldsymbol{\Psi}}_{(-)aL}\gamma_A\boldsymbol{\Psi}_{(-)bL}\right)$$

$$-\frac{1}{4\sqrt{30}}\left(i\mathbf{A}^{ij\dagger}_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)bij} - \overline{\boldsymbol{\Psi}}^{ij}_{(-)aL}\gamma_A\boldsymbol{\Psi}_{(-)bijL}\right)$$

$$+\frac{1}{2\sqrt{30}}\left(i\mathbf{A}^\dagger_{(-)ai}\overleftrightarrow{\partial}_A \mathbf{A}^i_{(-)b} + i\overline{\boldsymbol{\Psi}}_{(-)aiL}\gamma_A\boldsymbol{\Psi}^i_{(-)bL}\right)]\mathcal{V}'^A$$

$$+[\frac{1}{\sqrt{6}}\left(i\mathbf{A}^\dagger_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}^i_{(-)b} - \overline{\boldsymbol{\Psi}}_{(-)aL}\gamma_A\boldsymbol{\Psi}^i_{(-)bL}\right)]\mathcal{V}'^A_i$$

$$+[\frac{1}{\sqrt{6}}\left(i\mathbf{A}^\dagger_{(-)ai}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)b} - \overline{\boldsymbol{\Psi}}_{(-)aiL}\gamma_A\boldsymbol{\Psi}_{(-)bL}\right)]\mathcal{V}'^{Ai}$$

$$+[\frac{1}{4\sqrt{2}}\left(\mathbf{A}^\dagger_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)blm} + i\overline{\boldsymbol{\Psi}}_{(-)aL}\gamma_A\boldsymbol{\Psi}_{(-)blmL}\right)$$

$$-\frac{1}{24\sqrt{2}}\epsilon_{ijklm}\left(i\mathbf{A}^{ij\dagger}_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}^k_{(-)b} - \overline{\boldsymbol{\Psi}}^{ij}_{(-)aL}\gamma_A\boldsymbol{\Psi}^k_{(-)bL}\right)]\mathcal{V}'^{Alm}$$

$$+[\frac{1}{4\sqrt{2}}\left(i\mathbf{A}^{lm\dagger}_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)b} - \overline{\boldsymbol{\Psi}}^{lm}_{(-)aL}\gamma_A\boldsymbol{\Psi}_{(-)bL}\right)$$

$$-\frac{1}{24\sqrt{2}}\epsilon^{ijklm}\left(i\mathbf{A}^\dagger_{(-)ai}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)bjk} - \overline{\boldsymbol{\Psi}}_{(-)aiL}\gamma_A\boldsymbol{\Psi}_{(-)bjkL}\right)]\mathcal{V}'^A_{lm}$$

$$+[\frac{1}{6\sqrt{2}}\left(i\mathbf{A}^{jk\dagger}_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)bki} - \overline{\boldsymbol{\Psi}}^{jk}_{(-)aL}\gamma_A\boldsymbol{\Psi}_{(-)bkiL}\right)$$

$$-\frac{1}{2\sqrt{2}}\left(i\mathbf{A}^\dagger_{(-)ai}\overleftrightarrow{\partial}_A \mathbf{A}^j_{(-)b} - \overline{\boldsymbol{\Psi}}_{(-)aiL}\gamma_A\boldsymbol{\Psi}^j_{(-)bL}\right)]\mathcal{V}'^{iA}_j$$

$$+[\frac{1}{12\sqrt{6}}\epsilon_{ijklm}\left(i\mathbf{A}^{ij\dagger}_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}^n_{(-)b} - \overline{\boldsymbol{\Psi}}^{ij}_{(-)aL}\gamma_A\boldsymbol{\Psi}^n_{(-)bL}\right)]\mathcal{V}'^{Aklm}_n$$

$$+[\frac{1}{12\sqrt{6}}\epsilon^{ijklm}\left(i\mathbf{A}^\dagger_{(-)an}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)bij} - \overline{\boldsymbol{\Psi}}_{(-)anL}\gamma_A+\boldsymbol{\Psi}_{(-)bijL}\right)]\mathcal{V}'^{An}_{klm}$$

$$+[-\frac{1}{4\sqrt{6}}\epsilon_{ijklm}\left(i\mathbf{A}^{kl\dagger}_{(-)a}\overleftrightarrow{\partial}_A \mathbf{A}_{(-)bij} - \overline{\boldsymbol{\Psi}}^{kl}_{(-)aL}\gamma_A\boldsymbol{\Psi}_{(-)bijL}\right)]\mathcal{V}'^{Aij}_{kl}$$

$$-\frac{i}{2}\sqrt{\frac{5}{6}}\left[-\mathbf{A}^\dagger_{(-)a}\overline{\boldsymbol{\Psi}}_{(-)bR} + \frac{1}{10}\mathbf{A}^{ij\dagger}_{(-)a}\overline{\boldsymbol{\Psi}}_{(-)bijR} - \frac{1}{5}\mathbf{A}^\dagger_{(-)ai}\overline{\boldsymbol{\Psi}}^i_{(-)bR}\right]\Lambda'_L$$

$$+\frac{i}{\sqrt{3}}\left[\mathbf{A}^\dagger_{(-)a}\overline{\boldsymbol{\Psi}}^i_{(-)bR}\right]\Lambda'_{iL} + \frac{i}{\sqrt{3}}\left[\mathbf{A}^\dagger_{(-)ai}\overline{\boldsymbol{\Psi}}_{(-)bR}\right]\Lambda'^i_L$$

$$+\frac{i}{4}\left[\mathbf{A}^\dagger_{(-)a}\overline{\boldsymbol{\Psi}}_{(-)blmR} - \frac{1}{6}\epsilon_{ijklm}\mathbf{A}^{ij\dagger}_{(-)a}\overline{\boldsymbol{\Psi}}^k_{(-)bR}\right]\Lambda'^{lm}_L$$

6

$$+\frac{i}{4}\left[\mathbf{A}^{lm\dagger}_{(-)a}\overline{\boldsymbol{\Psi}}_{(-)bR}-\frac{1}{6}\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(-)ai}\overline{\boldsymbol{\Psi}}_{(-)bjkR}\right]\Lambda'_{lmL}$$

$$-\frac{i}{2}\left[-\frac{1}{3}\mathbf{A}^{jk\dagger}_{(-)a}\overline{\boldsymbol{\Psi}}_{(-)bkiR}+\mathbf{A}^{\dagger}_{(-)ai}\overline{\boldsymbol{\Psi}}^{j}_{(-)bR}\right]\Lambda'^{i}_{jL}$$

$$+\frac{i}{6\sqrt{3}}\epsilon_{ijklm}\left[\mathbf{A}^{ij\dagger}_{(-)a}\overline{\boldsymbol{\Psi}}^{n}_{(-)bR}\right]\Lambda'^{klm}_{nL}+\frac{i}{6\sqrt{3}}\epsilon^{ijklm}\left[\mathbf{A}^{\dagger}_{(-)an}\overline{\boldsymbol{\Psi}}_{(-)bijR}\right]\Lambda'^{n}_{klmL}$$

$$-\frac{i}{4\sqrt{3}}\left[\mathbf{A}^{kl\dagger}_{(-)a}\overline{\boldsymbol{\Psi}}_{(-)bijR}\right]\Lambda'^{ij}_{klL}$$

$$+\frac{i}{2}\sqrt{\frac{5}{6}}\left[-\overline{\boldsymbol{\Psi}}_{(-)aL}\mathbf{A}_{(-)b}+\frac{1}{10}\overline{\boldsymbol{\Psi}}^{ij}_{(-)aL}\mathbf{A}_{(-)bij}-\frac{1}{5}\overline{\boldsymbol{\Psi}}_{(-)aiL}\mathbf{A}^{i}_{(-)b}\right]\Lambda'_{R}$$

$$-\frac{i}{\sqrt{3}}\left[\overline{\boldsymbol{\Psi}}_{(-)aL}\mathbf{A}^{i}_{(-)b}\right]\Lambda'_{iR}-\frac{i}{\sqrt{3}}\left[\overline{\boldsymbol{\Psi}}_{(-)aiL}\mathbf{A}_{(-)b}\right]\Lambda'^{i}_{R}$$

$$-\frac{i}{4}\left[\overline{\boldsymbol{\Psi}}_{(-)aL}\mathbf{A}_{(-)blm}-\frac{1}{6}\epsilon_{ijklm}\overline{\boldsymbol{\Psi}}^{ij}_{(-)aL}\mathbf{A}^{k}_{(-)b}\right]\Lambda'^{lm}_{R}$$

$$-\frac{i}{4}\left[\overline{\boldsymbol{\Psi}}^{lm}_{(-)aL}\mathbf{A}_{(-)b}-\frac{1}{6}\epsilon^{ijklm}\overline{\boldsymbol{\Psi}}_{(-)aiL}\mathbf{A}_{(-)bjk}\right]\Lambda'_{lmR}$$

$$+\frac{i}{2}\left[-\frac{1}{3}\overline{\boldsymbol{\Psi}}^{jk}_{(-)aL}\mathbf{A}_{(-)bki}+\overline{\boldsymbol{\Psi}}_{(-)aiL}\mathbf{A}^{j}_{(-)b}\right]\Lambda'^{i}_{jR}$$

$$-\frac{i}{6\sqrt{3}}\epsilon_{ijklm}\left[\overline{\boldsymbol{\Psi}}^{ij}_{(-)aL}\mathbf{A}^{n}_{(-)b}\right]\Lambda'^{klm}_{nR}-\frac{i}{6\sqrt{3}}\epsilon^{ijklm}\left[\overline{\boldsymbol{\Psi}}_{(-)anL}\mathbf{A}_{(-)bij}\right]\Lambda'^{n}_{klmR}$$

$$+\frac{i}{4\sqrt{3}}\left[\overline{\boldsymbol{\Psi}}^{kl}_{(-)aL}\mathbf{A}_{(-)bij}\right]\Lambda'^{ij}_{klR}\}+\mathsf{L}^{(210)}_{(4)auxiliary}$$

$$\mathsf{L}^{(210)}_{(4)auxiliary}=\frac{h^{(210-)}_{ab}\mathsf{g}^{(210)}}{4!2}<A_{(-)a}|\Gamma_{[\mu}\Gamma_{\nu}\Gamma_{\rho}\Gamma_{\lambda]}|A_{(-)b}>D_{\mu\nu\rho\lambda} \qquad (8)$$

Further, the kinetic energy for the vector multiplet is given by

$$\mathsf{L}^{(210\ K.E.)}_{V}=\frac{1}{64}\left[\widehat{\mathcal{W}}^{\tilde{\alpha}}_{\mu\nu\rho\lambda}\widehat{\mathcal{W}}_{\tilde{\alpha}\mu\nu\rho\lambda}|_{\theta^2}+\widehat{\overline{\mathcal{W}}}_{\dot{\tilde{\alpha}}\mu\nu\rho\lambda}\widehat{\overline{\mathcal{W}}}^{\dot{\tilde{\alpha}}}_{\mu\nu\rho\lambda}|_{\bar{\theta}^2}\right]$$

$$\widehat{\mathcal{W}}^{\tilde{\alpha}}_{\mu\nu\rho\lambda}=\overline{\mathsf{D}}^2\mathsf{D}_{\tilde{\alpha}}\widehat{\mathsf{V}}_{\mu\nu\rho\lambda},\quad \widehat{\overline{\mathcal{W}}}_{\dot{\tilde{\alpha}}\mu\nu\rho\lambda}=\mathsf{D}^2\overline{\mathsf{D}}_{\dot{\tilde{\alpha}}}\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}$$

$$\mathsf{D}_{\tilde{\alpha}}=\frac{\partial}{\partial\theta^{\tilde{\alpha}}}+i\sigma^{A}_{\tilde{\alpha}\dot{\tilde{\alpha}}}\overline{\theta}^{\dot{\tilde{\alpha}}}\partial_{A};\quad \overline{\mathsf{D}}_{\dot{\tilde{\alpha}}}=-\frac{\partial}{\partial\overline{\theta}^{\dot{\tilde{\alpha}}}}-i\theta^{\tilde{\alpha}}\sigma_{\tilde{\alpha}\dot{\tilde{\alpha}}} \qquad (9)$$

Explicit evaluation of Eq.(1) gives

$$\mathsf{L}^{(210\ K.E.)}_{V}=-\frac{1}{4}\mathcal{V}_{AB\mu\nu\rho\sigma}\mathcal{V}^{AB}_{\mu\nu\rho\sigma}-\frac{i}{2}\overline{\Lambda}_{\mu\nu\rho\sigma}\gamma^{A}\partial_{A}\Lambda_{\mu\nu\rho\sigma}+\mathsf{L}^{(210)}_{(5)auxiliary}$$

$$\mathcal{V}^{AB}_{\mu\nu\rho\sigma}=\partial^{A}\mathcal{V}^{B}_{\mu\nu\rho\sigma}-\partial^{B}\mathcal{V}^{A}_{\mu\nu\rho\sigma}$$

$$\mathsf{L}^{(210)}_{(5)auxiliary}=\frac{1}{2}D_{\mu\nu\rho\sigma}D_{\mu\nu\rho\sigma}$$

$$\Lambda_{\mu\nu\rho\sigma}=\begin{pmatrix}\lambda_{\tilde{\alpha}\mu\nu\rho\sigma}\\\overline{\lambda}^{\dot{\tilde{\alpha}}}_{\mu\nu\rho\sigma}\end{pmatrix} \qquad (10)$$

Finally, the superpotential of the theory is taken to be

$$\mathsf{L}^{(210)}_{\mathsf{W}}=\mathsf{W}(\widehat{\Phi}_{(+)},\widehat{\Phi}_{(-)})|_{\theta^2}+h.c.$$

$$W(\widehat{\Phi}_{(+)}, \widehat{\Phi}_{(-)}) = \mu_{ab} < \widehat{\Phi}^*_{(-)a}|B|\widehat{\Phi}_{(+)b} >$$

$$W(\mathbf{A}_{(+)}, \mathbf{A}_{(-)}) = i\mu_{ab}\left(\mathbf{A}^{\mathbf{T}}_{(-)a}\mathbf{A}_{(+)b} - \frac{1}{2}\mathbf{A}^{\mathbf{T}}_{(-)aij}\mathbf{A}^{ij}_{(+)b} + \mathbf{A}^{i\mathbf{T}}_{(-)a}\mathbf{A}_{(+)bi}\right) \qquad (11)$$

where μ_{ab} is taken to be a symmetric tensor and B is the usual SO(10)charge conjugation operator . Thus we have

$$\mathsf{L_W} = -i\mu_{ab}\left(\overline{\boldsymbol{\Psi}}_{(-)aR}\boldsymbol{\Psi}_{(+)bL} + \overline{\boldsymbol{\Psi}}^i_{(-)aR}\boldsymbol{\Psi}_{(+)biL} - \frac{1}{2}\overline{\boldsymbol{\Psi}}_{(-)aijR}\boldsymbol{\Psi}^{ij}_{(+)bL}\right)$$

$$+i\mu^*_{ab}\left(\overline{\boldsymbol{\Psi}}_{(-)aL}\boldsymbol{\Psi}_{(+)bR} + \overline{\boldsymbol{\Psi}}^i_{(-)aiL}\boldsymbol{\Psi}^i_{(+)bR} - \frac{1}{2}\overline{\boldsymbol{\Psi}}^{ij}_{(-)aL}\boldsymbol{\Psi}_{(+)bijR}\right) + \mathsf{L}^{(210)}_{(6)auxiliary}$$

$$\mathsf{L}^{(210)}_{(6)auxiliary} = i\mu_{ab}[\mathbf{F}_{(-)a}\mathbf{A}_{(+)b} + \mathbf{A}^{\mathbf{T}}_{(-)a}\mathbf{F}_{(+)b} - \frac{1}{2}\mathbf{F}_{(-)aij}\mathbf{A}^{ij}_{(+)b}$$

$$-\frac{1}{2}\mathbf{A}^{\mathbf{T}}_{(-)aij}\mathbf{F}^{ij}_{(+)b} + \mathbf{F}^i_{(-)a}\mathbf{A}_{(+)bi} + \mathbf{A}^{i\mathbf{T}}_{(-)a}\mathbf{F}_{(+)bi}] + h.c. \quad (12)$$

Elimination of the auxiliary fields is carried out in appendix C.

3 A more general analysis of $16_+ - \overline{16}_+ - 210$ vector couplings

In this section we consider the couplings of the unconstrained 210 vector multiplet with matter, i.e., in the analysis we use the full vector multiplet rather than the truncated one under the constraint of the Wess-Zumino gauge. An illustration of this procedure is given in Appendix G for the $U(1)$ case. The Lagrangian that governs the interactions of the 210 multiplet consists of the kinetic energy term for the 210 plet, self interactions, and interactions of the 210 plet with 16 and $\overline{16}$ of matter. For generality we also include a mass term for the 210 vector multiplet. As in Appendix G we will not impose the Wess-Zumino gauge but keep the full multiplet. Thus we take the Lagrangian governing the 210 vector multiplet to be

$$\mathsf{L}^{(210)} = \mathsf{L}^{\prime(210\ K.E.)}_V + \mathsf{L}^{\prime(210\ Mass)}_V + \mathsf{L}^{\prime(210\ Self-Interaction)}_V + \mathsf{L}^{\prime(210\ Interaction)}_{V+\Phi} + \mathsf{L}^{\prime(210\ Self-Interaction)}_\Phi$$

$$\mathsf{L}^{\prime(210\ K.E.)}_V = \frac{1}{64}\left[\widehat{\mathcal{W}}^{\tilde{\alpha}}_{\mu\nu\rho\lambda}\widehat{\mathcal{W}}_{\tilde{\alpha}\mu\nu\rho\lambda}|_{\theta^2} + \widehat{\overline{\mathcal{W}}}_{\dot{\tilde{\alpha}}\mu\nu\rho\lambda}\widehat{\overline{\mathcal{W}}}^{\dot{\tilde{\alpha}}}_{\mu\nu\rho\lambda}|_{\bar{\theta}^2}\right]$$

$$\mathsf{L}^{\prime(210\ Mass)}_V = m^2\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}|_{\theta^2\bar{\theta}^2}$$

$$\mathsf{L}^{\prime(210\ Self-Interaction)}_V = \alpha_1\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}\widehat{\mathsf{V}}_{\rho\lambda\alpha\beta}\widehat{\mathsf{V}}_{\alpha\beta\mu\nu}|_{\theta^2\bar{\theta}^2} + \alpha_2\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}\widehat{\mathsf{V}}_{\rho\lambda\alpha\beta}\widehat{\mathsf{V}}_{\alpha\beta\eta\tau}\widehat{\mathsf{V}}_{\eta\tau\mu\nu}|_{\theta^2\bar{\theta}^2}$$

$$\mathsf{L}^{\prime(210\ Interaction)}_{V+\Phi} = \frac{h^{(210)}_{ab}}{4!}\widehat{\Phi}^\dagger_a\widehat{\mathsf{V}}_{\mu\nu\rho\lambda}\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}\widehat{\Phi}_b|_{\theta^2\bar{\theta}^2}$$

$$\widehat{\mathsf{L}}^{\prime(210\ Self-Interaction)}_\Phi = \widehat{\Phi}^\dagger_a\widehat{\Phi}_a|_{\theta^2\bar{\theta}^2} \qquad (13)$$

8

In the above the Greek subscripts(α, β, ...) are the SO(10) indices. One could, of course, add more interactions, for example, in $\mathsf{L}_V^{'(210\ Self-Interaction)}$ such as V^5 etc which are allowed once one gives up the Wess-Zumino gauge. Similarly in $\mathsf{L}_{V+\Phi}^{'(210\ Interaction)}$ one may add additional terms as well. However, the line of construction remains unchanged and the inclusion of additional terms only brings in more complexity. Thus to keep the analysis simple we omit such terms. Evaluating Eq.(13) we get

$$
\begin{aligned}
\mathsf{L}^{(210)} =\ & -\frac{1}{4}\mathcal{V}_{AB\mathsf{XY}}\mathcal{V}_{\mathsf{XY}}^{AB} - \frac{1}{2}m^2\mathcal{V}_{A\mathsf{XY}}\mathcal{V}_{\mathsf{XY}}^{A} - \frac{1}{2}\partial^A B_{\mathsf{XY}}\partial_A B_{\mathsf{XY}} - i\overline{\Lambda}_{\mathsf{XY}}\gamma^A\partial_A\Lambda_{\mathsf{XY}} - m\overline{\Lambda}_{\mathsf{XY}}\Lambda_{\mathsf{XY}} \\
& -\partial^A A_a^\dagger\partial_A A_a - i\overline{\Psi}_{aL}\gamma^A\partial_A\Psi_{aL} - \frac{h_{ab}^{(210)}}{24m}B_{\mathsf{XY}}(\partial^A A_a^\dagger)\widetilde{\Gamma}_{\mathsf{XY}}\partial_A A_b \\
& -\frac{h_{ab}^{(210)}}{96m}\partial^A\left(A_a^\dagger\widetilde{\Gamma}_{\mathsf{XY}}A_b\right)\partial_A B_{\mathsf{XY}} + \frac{ih_{ab}^{(210)}}{48}\left[A_a^\dagger\widetilde{\Gamma}_{\mathsf{XY}}\partial^A A_b - (\partial^A A_a^\dagger)\widetilde{\Gamma}_{\mathsf{XY}}A_b\right]\mathcal{V}_{A\mathsf{XY}} \\
& +\frac{h_{ab}^{(210)}}{48m\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\gamma^A\widetilde{\Gamma}_{\mathsf{XY}}\Lambda_{L\mathsf{XY}}\right)\partial_A A_b + \partial_A A_a^\dagger\left(\overline{\Lambda}_{L\mathsf{XY}}\gamma^A\widetilde{\Gamma}_{\mathsf{XY}}\Psi_{bL}\right)\right] \\
& +\frac{h_{ab}^{(210)}}{24m\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\gamma^A\widetilde{\Gamma}_{\mathsf{XY}}\partial_A\Lambda_{L\mathsf{XY}}\right)A_b - A_a^\dagger\left(\overline{\Lambda}_{L\mathsf{XY}}\gamma^A\widetilde{\Gamma}_{\mathsf{XY}}\partial_A\Psi_{bL}\right)\right] \\
& -\frac{ih_{ab}^{(210)}}{24\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\widetilde{\Gamma}_{\mathsf{XY}}\Lambda_{R\mathsf{XY}}\right)A_b - A_a^\dagger\left(\overline{\Lambda}_{R\mathsf{XY}}\widetilde{\Gamma}_{\mathsf{XY}}\Psi_{bL}\right)\right] \\
& -\frac{h_{ab}^{(210)}}{48}\overline{\Psi}_{aL}\gamma^A\widetilde{\Gamma}_{\mathsf{XY}}\Psi_{bL}\mathcal{V}_{A\mathsf{XY}} - \frac{ih_{ab}^{(210)}}{24m}\overline{\Psi}_{aL}\gamma^A\widetilde{\Gamma}_{\mathsf{XY}}\partial_A(\Psi_{bL})B_{\mathsf{XY}} \\
& -\frac{1}{m^3}\left(\frac{3\alpha_1}{4}B_{\mathsf{WY}} + \frac{\alpha_2}{m}B_{\mathsf{WX}}B_{\mathsf{XY}}\right)\partial_A B_{\mathsf{YZ}}\partial^A B_{\mathsf{ZW}} - \frac{3}{m}\left(\frac{\alpha_1}{2}B_{\mathsf{WY}} + \frac{\alpha_2}{m}B_{\mathsf{WX}}B_{\mathsf{XY}}\right)\mathcal{V}_{A\mathsf{YZ}}\mathcal{V}_{\mathsf{ZW}}^A \\
& +\frac{3}{m^3}\left(\alpha_1 B_{\mathsf{WY}} + \frac{2\alpha_2}{m}B_{\mathsf{WX}}B_{\mathsf{XY}}\right)\left(i\overline{\Lambda}_{L\mathsf{YZ}}\gamma^A\partial_A\Lambda_{L\mathsf{ZW}} - m\overline{\Lambda}_{\mathsf{YZ}}\Lambda_{\mathsf{ZW}}\right) \\
& +\frac{3}{m^2}\left(\frac{\alpha_1}{2}\delta_{\mathsf{WX}} + \frac{2\alpha_2}{m}B_{\mathsf{WX}}\right)\left(\overline{\Lambda}_{L\mathsf{XY}}\gamma^A\Lambda_{L\mathsf{YZ}}\right)\mathcal{V}_{A\mathsf{ZW}} \\
& +\frac{3\alpha_2}{2m^4}\left(\overline{\Lambda}_{R\mathsf{WX}}^c\Lambda_{L\mathsf{XY}}\right)\left(\overline{\Lambda}_{L\mathsf{YZ}}\Lambda_{R\mathsf{ZW}}^c\right) + \mathsf{L}_{auxiliary}^{'\ 210} \qquad (14)
\end{aligned}
$$

Where we have defined for brevity

$$
\widetilde{\Gamma}_{\mathsf{XY}} = \Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]};\quad B_{\mathsf{XY}} = B_{\mu\nu\rho\sigma};\quad \mathcal{V}_{A\mathsf{XY}}\mathcal{V}_{\mathsf{YZ}}^A = \mathcal{V}_{A\mu\nu\rho\sigma}\mathcal{V}_{\rho\sigma\lambda\tau}^A \qquad (15)
$$

and so on. Further

$$
\Lambda = \begin{pmatrix} m\chi_{\tilde{\alpha}} \\ \overline{\lambda}^{\dot{\tilde{\alpha}}} \end{pmatrix},\quad \Psi_a = \begin{pmatrix} \psi_{a\tilde{\alpha}} \\ \overline{\psi}_a^{\dot{\tilde{\alpha}}} \end{pmatrix},\quad B = mC,
$$

$$
\Lambda^c = \mathcal{C}\overline{\Lambda}^T,\quad \mathcal{C} = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\overline{\sigma}^2 \end{pmatrix},\quad \overline{\Lambda} = \Lambda^\dagger\gamma^0 \qquad (16)
$$

9

Note that each of the chiral fields, S ($\equiv A, \Psi, F$) can be expanded in terms of its SU(5) components as $|S_a> = |0> \mathbf{S}_a + \frac{1}{2} b_i^\dagger b_j^\dagger |0> \mathbf{S}_a^{ij} + \frac{1}{24} \epsilon^{ijklm} b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger |0> \mathbf{S}_{ai}$. We will expand some of the terms appearing in Eq.(14) in appendix D.

The Lagrangian containing the auxiliary fields is given by

$$
\mathsf{L}_{auxilliary}^{'\,210} = \left(m B_{\mathtt{ZW}} + \frac{3\alpha_1}{2m^2} B_{\mathtt{WX}} B_{\mathtt{XZ}} + \frac{2\alpha_2}{m^3} B_{\mathtt{WX}} B_{\mathtt{XY}} B_{\mathtt{YZ}} + \frac{h_{ab}^{(210)}}{48} A_a^\dagger \widetilde{\Gamma}_{\mathtt{ZW}} A_b \right) D_{\mathtt{ZW}} + \frac{1}{2} D_{\mathtt{XY}} D_{\mathtt{XY}}
$$

$$
+ \left(\frac{1}{2} m^2 \delta_{\mathtt{YW}} + \frac{3\alpha_1}{2m} B_{\mathtt{YW}} + \frac{3\alpha_2}{m^2} B_{\mathtt{WX}} B_{\mathtt{XY}} \right) (M_{\mathtt{YZ}} M_{\mathtt{ZW}} + N_{\mathtt{YZ}} N_{\mathtt{ZW}})
$$

$$
+ i \left[\frac{h_{ab}^{(210)}}{48} F_a^\dagger \widetilde{\Gamma}_{\mathtt{ZW}} A_b + \frac{3}{m^2} \left(\frac{\alpha_1}{4} \delta_{\mathtt{WX}} + \frac{\alpha_2}{m} B_{\mathtt{WX}} \right) \left(\overline{\Lambda}_{L\mathtt{XY}} \Lambda_{R\mathtt{YZ}}^c \right) \right] (M_{\mathtt{ZW}} + i N_{\mathtt{ZW}})
$$

$$
- i \left[\frac{h_{ab}^{(210)}}{48} A_a^\dagger \widetilde{\Gamma}_{\mathtt{ZW}} F_b + \frac{3}{m^2} \left(\frac{\alpha_1}{4} \delta_{\mathtt{WX}} + \frac{\alpha_2}{m} B_{\mathtt{WX}} \right) \left(\overline{\Lambda}_{R\mathtt{XY}}^c \Lambda_{L\mathtt{YZ}} \right) \right] (M_{\mathtt{ZW}} - i N_{\mathtt{ZW}})
$$

$$
+ \frac{i h_{ab}^{(210)}}{24 m \sqrt{2}} \left(\overline{\Lambda}_{L\mathtt{XY}} \Psi_{aR} \widetilde{\Gamma}_{\mathtt{XY}} \right) F_b - \frac{i h_{ba}^{(210)}}{24 m \sqrt{2}} \left(\widetilde{\Gamma}_{\mathtt{XY}} \overline{\Psi}_{aR} \Lambda_{L\mathtt{XY}} \right) F_b^\dagger
$$

$$
+ \frac{h_{ab}^{(210)}}{24 m} B_{\mathtt{XY}} F_a^\dagger \widetilde{\Gamma}_{\mathtt{XY}} F_b + F_a^\dagger F_a \quad (17)
$$

Finally, eliminating the auxiliary fields we obtain

$$
\mathsf{L}_{auxilliary}^{'} = -\frac{1}{2} m^2 B_{\mathtt{XY}} B_{\mathtt{YX}} - \frac{3\alpha_1}{2m} B_{\mathtt{XY}} B_{\mathtt{YZ}} B_{\mathtt{ZX}} - \left(\frac{9\alpha_1^2}{8m^4} + \frac{2\alpha_2}{m^2} \right) B_{\mathtt{WX}} B_{\mathtt{XY}} B_{\mathtt{YZ}} B_{\mathtt{ZW}}
$$

$$
- \frac{3\alpha_1 \alpha_2}{m^5} B_{\mathtt{VW}} B_{\mathtt{WX}} B_{\mathtt{XY}} B_{\mathtt{YZ}} B_{\mathtt{ZV}} - \frac{2\alpha_2^2}{m^6} B_{\mathtt{UV}} B_{\mathtt{VW}} B_{\mathtt{WX}} B_{\mathtt{XY}} B_{\mathtt{YZ}} B_{\mathtt{ZU}}
$$

$$
- \frac{h_{ab}^{(210)} h_{cd}^{(210)}}{4608} \left(A_a^\dagger \widetilde{\Gamma}_{\mathtt{XY}} A_b \right) \left(A_c^\dagger \widetilde{\Gamma}_{\mathtt{XY}} A_d \right) - \frac{m h_{ab}^{(210)}}{48} \left(A_a^\dagger \widetilde{\Gamma}_{\mathtt{XY}} A_b \right) B_{\mathtt{XY}}
$$

$$
- \frac{h_{ab}^{(210)} \alpha_1}{32 m^2} \left(A_a^\dagger \widetilde{\Gamma}_{\mathtt{XY}} A_b \right) B_{\mathtt{XZ}} B_{\mathtt{ZY}} - \frac{h_{ab}^{(210)} \alpha_2}{24 m^3} \left(A_a^\dagger \widetilde{\Gamma}_{\mathtt{XY}} A_b \right) B_{\mathtt{XW}} B_{\mathtt{WZ}} B_{\mathtt{ZY}}
$$

$$
- \frac{1}{2} m^2 \left[\mathbf{K}_{\mathtt{WX}} (\mathbf{P}^{-1})_{\mathtt{WXYZ}} \mathbf{K}_{\mathtt{YZ}} + \mathbf{J}_{\mathtt{WX}} (\mathbf{P}^{-1})_{\mathtt{WXYZ}} \mathbf{J}_{\mathtt{YZ}} \right] - \frac{i h_{ab}^{(210)}}{24 m \sqrt{2}} (\mathbf{Q}^{-1})_{ac} \mathbf{R}_c \widetilde{\Gamma}_{\mathtt{XY}} \overline{\Psi}_{bR} \Lambda_{L\mathtt{XY}}
$$

$$
+ \frac{h_{ab}^{(210)}}{8 m^4} A_a^\dagger \widetilde{\Gamma}_{\mathtt{UV}} (\mathbf{S}^{-1})_{bc} \mathbf{T}_c (\mathbf{P}^{-1})_{\mathtt{UVXY}} \left(\frac{\alpha_1}{4} \delta_{\mathtt{YW}} + \frac{\alpha_2}{m} B_{\mathtt{YW}} \right) \overline{\Lambda}_{L\mathtt{WZ}} \Lambda_{R\mathtt{ZX}}^c \quad (18)
$$

where

$$
\mathbf{P}_{\mathtt{UVXY}} = \delta_{\mathtt{UX}} \delta_{\mathtt{VY}} + \frac{3\alpha_1}{2m^3} (\delta_{\mathtt{UY}} B_{\mathtt{VX}} + B_{\mathtt{UX}} \delta_{\mathtt{VY}}) + \frac{3\alpha_2}{m^4} (\delta_{\mathtt{UY}} B_{\mathtt{VW}} B_{\mathtt{WX}} + B_{\mathtt{UW}} B_{\mathtt{WY}} \delta_{\mathtt{XV}})
$$

$$
\mathbf{Q}_{bc} = \delta_{bc} + \frac{h_{bc}^{(210)}}{24 m} \widetilde{\Gamma}_{\mathtt{XY}} B_{\mathtt{XY}} - \frac{h_{ac}^{(210)} h_{bd}^{(210)}}{1152 m^2} A_a^\dagger \widetilde{\Gamma}_{\mathtt{UV}} (\mathbf{P}^{-1})_{\mathtt{UVXY}} \widetilde{\Gamma}_{\mathtt{XY}} A_d
$$

$$
\mathbf{S}_{bc} = \delta_{bc} + \frac{h_{cb}^{(210)}}{24 m} \widetilde{\Gamma}_{\mathtt{XY}} B_{\mathtt{XY}} - \frac{h_{cd}^{(210)} h_{ab}^{(210)}}{1152 m^2} \widetilde{\Gamma}_{\mathtt{UV}} A_d (\mathbf{P}^{-1})_{\mathtt{UVXY}} A_a^\dagger \widetilde{\Gamma}_{\mathtt{XY}}
$$

10

$$\mathbf{R}_c = -\frac{ih_{ac}^{(210)}}{24m\sqrt{2}}\overline{\Lambda}_{LXY}\Psi_{aR}\widetilde{\Gamma}_{XY} + \frac{h_{ac}^{(210)}}{8m^4}A_a^\dagger\widetilde{\Gamma}_{UV}(\mathbf{P}^{-1})_{UVXY}\left(\frac{\alpha_1}{4}\delta_{YW} + \frac{\alpha_2}{m}B_{YW}\right)\overline{\Lambda}_{LWZ}\Lambda_{RZX}^c$$

$$\mathbf{T}_c = \frac{ih_{ca}^{(210)}}{24m\sqrt{2}}\widetilde{\Gamma}_{XY}\overline{\Psi}_{aR}\Lambda_{LXY} + \frac{h_{ca}^{(210)}}{8m^4}\widetilde{\Gamma}_{UV}A_a(\mathbf{P}^{-1})_{UVXY}\left(\frac{\alpha_1}{4}\delta_{YW} + \frac{\alpha_2}{m}B_{YW}\right)\overline{\Lambda}_{RWZ}^c\Lambda_{LZX}$$

$$\mathbf{K}_{XY} = -\frac{ih_{ab}^{(210)}}{48m^2}\left[(\mathbf{Q}^{-1})_{ac}\mathbf{R}_c\widetilde{\Gamma}_{XY}A_b - A_a^\dagger\widetilde{\Gamma}_{XY}(\mathbf{S}^{-1})_{bc}\mathbf{T}_c\right]$$
$$-\frac{3i}{m^4}\left(\frac{\alpha_1}{4}\delta_{XU} + \frac{\alpha_2}{m}B_{XU}\right)\left(\overline{\Lambda}_{LUV}\Lambda_{RVY}^c - \overline{\Lambda}_{RUV}^c\Lambda_{LVY}\right)$$

$$\mathbf{J}_{XY} = \frac{h_{ab}^{(210)}}{48m^2}\left[(\mathbf{Q}^{-1})_{ac}\mathbf{R}_c\widetilde{\Gamma}_{XY}A_b + A_a^\dagger\widetilde{\Gamma}_{XY}(\mathbf{S}^{-1})_{bc}\mathbf{T}_c\right]$$
$$+\frac{3}{m^4}\left(\frac{\alpha_1}{4}\delta_{XU} + \frac{\alpha_2}{m}B_{XU}\right)\left(\overline{\Lambda}_{LUV}\Lambda_{RVY}^c + \overline{\Lambda}_{RUV}^c\Lambda_{LVY}\right)(19)$$

4   Conclusion

In this paper we have given an explicit computation of the couplings of the 210 dimensional SO(10) vector mutliplet. Specifically, we have computed the vector couplings $\overline{16}_\pm - 16_\pm - 210$ in terms of its $SU(5) \times U(1)$ decomposition. We approached this coupling from two view points. First, we use the conventional approach of using the Wess-Zumino gauge. However, since the 210 couplings are not expected to be gauge invariant and hence such interactions are not expected to be renormalizable, we also consider a nonlinear sigma model type couplings of 210 with matter. Such couplings arise when we consider the full 210 multiplet without using the Wess-Zumino gauge. Here elimination of the auxiliary fields leads to interactions of the vector multiplet with the chiral fields with nonlinearities of infinite order as in a nonlinear sigma model. Although couplings of the type discussed do not thus far appear in theories of fundamental interactions a 210 vector multiplet may arise as a condensate in effective theories. The analysis we have presented here concludes our effort to give a complete analytic computation of all lowest order couplings involving the $16_\pm$ of matter with Higgs and vector multiplets. Although the analysis given here is specific to the case of the vector couplings $\overline{16}_\pm - 16_\pm - 210$ the techniques developed here are general and can be applied to other cases where the dimensionality of the vector mulitiplet does not equal the dimensionality of the adjoint representation of the group.

5 Appendix A

In this section we define the notation for the components of the vector superfield and for the chiral superfield used in the text. For the vector superfield we have the expansion

$$\hat{V} = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta^2\left[M(x) + iN(x)\right] - \frac{i}{2}\bar{\theta}^2\left[M(x) - iN(x)\right] - \theta\sigma^A\bar{\theta}\mathcal{V}_A(x)$$

$$+i\theta^2\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^A\partial_A\chi(x)\right] - i\bar{\theta}^2\theta\left[\lambda(x) + \frac{i}{2}\sigma^A\partial_A\bar{\chi}(x)\right] + \frac{1}{2}\theta^2\bar{\theta}^2\left[D(x) + \frac{1}{2}\partial_A\partial^A C(x)\right] \tag{20}$$

while for the chiral superfields we have the expansion

$$\hat{\Phi}_a = A_a(x) + \sqrt{2}\theta\psi_a + \theta^2 F_a(x) + i\theta\sigma^A\bar{\theta}\partial_A A_a(x)$$
$$+ \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^A\partial_A\psi_a(x) + \frac{1}{4}\theta^2\bar{\theta}^2\partial_A\partial^A A_a(x)$$
$$\hat{\Phi}_a^\dagger = A_a^\dagger(x) + \sqrt{2}\bar{\theta}\bar{\psi}_a + \bar{\theta}^2 F_a^\dagger(x) - i\theta\sigma^A\bar{\theta}\partial_A A_a^\dagger(x)$$
$$+ \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^A\partial_A\bar{\psi}_a(x) + \frac{1}{4}\theta^2\bar{\theta}^2\partial_A\partial^A A_a^\dagger(x) \tag{21}$$

6 Appendix B

In this appendix we normalize the irreducible SU(5) tensors contained in a 210 vector $\mathcal{V}^A_{\mu\nu\rho\sigma}$, a 210 scalar $B_{\mu\nu\rho\sigma}$, and a 210 spinor $\Lambda_{\mu\nu\rho\sigma}$. Latin letters $(i, j, k, ...)$ are used to denote the SU(5) indices. The normalized SU(5) gauge tensors appearing in $\mathcal{V}^A_{\mu\nu\rho\sigma}$ are

$$\mathcal{V}_A = 4\sqrt{\frac{10}{3}}\mathcal{V}'_A; \quad \mathcal{V}^i_A = 8\sqrt{6}\mathcal{V}'^i_A; \quad \mathcal{V}_{Ai} = 8\sqrt{6}\mathcal{V}'_{Ai}$$

$$\mathcal{V}^{ij}_A = \sqrt{2}\mathcal{V}'^{ij}_A; \quad \mathcal{V}_{Aij} = \sqrt{2}\mathcal{V}'_{Aij}; \quad \mathcal{V}^j_{Ai} = \sqrt{2}\mathcal{V}'^j_{Ai}$$

$$\mathcal{V}^{ijk}_{Al} = \sqrt{\frac{2}{3}}\mathcal{V}'^{ijk}_{Al}; \quad \mathcal{V}^i_{Ajkl} = \sqrt{\frac{2}{3}}\mathcal{V}'^i_{Ajkl}; \quad \mathcal{V}^{ij}_{Akl} = \sqrt{\frac{2}{3}}\mathcal{V}'^{ij}_{Akl} \tag{22}$$

so that

$$-\frac{1}{4}\mathcal{V}^{AB}_{\mu\nu\rho\sigma}\mathcal{V}_{AB\mu\nu\rho\sigma} = -\frac{1}{2}\mathcal{V}'_{AB}\mathcal{V}'^{AB\dagger} - \frac{1}{2}\mathcal{V}'^i_{AB}\mathcal{V}'^{ABi\dagger} - \frac{1}{2!}\frac{1}{2}\mathcal{V}'^{ij}_{AB}\mathcal{V}'^{ABij\dagger}$$

$$-\frac{1}{4}\mathcal{V}'^i_{ABj}\mathcal{V}'^{ABj}_i - \frac{1}{3!}\frac{1}{2}\mathcal{V}'^{ijk}_{ABl}\mathcal{V}'^{ABijk\dagger}_l - \frac{1}{2!}\frac{1}{2!}\frac{1}{4}\mathcal{V}'^{ij}_{ABkl}\mathcal{V}'^{ABkl}_{ij} \tag{23}$$

12

The normalized SU(5) fields appearing in $B_{\mu\nu\rho\sigma}$ are

$$B = 4\sqrt{\frac{10}{3}}B'; \quad B^i = 8\sqrt{6}B'^i; \quad B_i = 8\sqrt{6}B'_i$$

$$B^{ij} = \sqrt{2}B'^{ij}; \quad B_{ij} = \sqrt{2}B'_{ij}; \quad B_i^j = \sqrt{2}B_i'^j$$

$$B_l^{ijk} = \sqrt{\frac{2}{3}}B_l'^{ijk}; \quad B_{jkl} = \sqrt{\frac{2}{3}}B_{jkl}'^i; \quad B_{kl}^{ij} = \sqrt{\frac{2}{3}}B_{kl}'^{ij} \tag{24}$$

so that

$$-\frac{1}{2}\partial^A B_{\mu\nu\rho\sigma}\partial_A B_{\mu\nu\rho\sigma} = -\partial^A B'\partial_A B'^\dagger - \partial^A B'^i\partial_A B'^{i\dagger} - \frac{1}{2!}\partial^A B'^{ij}\partial_A B'^{ij\dagger}$$

$$-\frac{1}{2}\partial^A B_j'^i\partial_A B_i'^j - \frac{1}{3!}\partial^A B_l'^{ijk}\partial_A B_l'^{ijk\dagger} - \frac{1}{2!}\frac{1}{2!}\frac{1}{2}\partial^A B_{kl}'^{ij}\partial_A B_{ij}'^{kl} \tag{25}$$

The normalized SU(5) fields appearing in $\Lambda_{\mu\nu\rho\sigma}$ are

$$\Lambda = 4\sqrt{\frac{5}{3}}\Lambda'; \quad \Lambda_i = 8\sqrt{6}\Lambda'^i; \quad \Lambda_i = 8\sqrt{6}\Lambda'_i$$

$$\Lambda^{ij} = \sqrt{2}\Lambda'^{ij}; \quad \Lambda_{ij} = \sqrt{2}\Lambda'_{ij}; \quad \Lambda_i^j = \sqrt{2}\Lambda_i'^j$$

$$\Lambda_l^{ijk} = \sqrt{\frac{2}{3}}\Lambda_l'^{ijk}; \quad \Lambda_{jkl} = \sqrt{\frac{2}{3}}\Lambda_{jkl}'^i; \quad \Lambda_{kl}^{ij} = \sqrt{\frac{2}{3}}\Lambda_{kl}'^{ij} \tag{26}$$

so that

$$-i\overline{\Lambda}_{\mu\nu\rho\sigma}\gamma^A\partial_A\Lambda_{\mu\nu\rho\sigma} = -i\overline{\Lambda}'\gamma^A\partial_A\Lambda' - i\overline{\Lambda}_i'\gamma^A\partial_A\Lambda_i' - i\overline{\Lambda}'^i\gamma^A\partial_A\Lambda'^i$$

$$-\frac{1}{2!}i\overline{\Lambda}_{ij}'\gamma^A\partial_A\Lambda_{ij}' - \frac{1}{2!}i\overline{\Lambda}'^{ij}\gamma^A\partial_A\Lambda'^{ij} - i\overline{\Lambda}_j'^i\gamma^A\partial_A\Lambda_j'^i$$

$$-\frac{1}{3!}i\overline{\Lambda}_l'^{ijk}\gamma^A\partial_A\Lambda_l'^{ijk} - \frac{1}{3!}i\overline{\Lambda}_{ijk}'^l\gamma^A\partial_A\Lambda_{ijk}'^l - \frac{1}{2!}\frac{1}{2!}i\overline{\Lambda}_{kl}'^{ij}\gamma^A\partial_A\Lambda_{kl}'^{ij} \tag{27}$$

7   Appendix C

In this appendix we eliminate the auxiliary fields, $D_{\mu\nu\rho\sigma}$ and $\mathbf{F}_{(\pm)}$ of section 2. We find

$$\mathsf{L}_{(3)auxiliary}^{(210)} + \mathsf{L}_{(4)auxiliary}^{(210)} + \mathsf{L}_{(5)auxiliary}^{(210)}$$

$$= -\frac{1}{4608}\mathsf{g}^{(210)2}h_{ab}^{(210+)}h_{cd}^{(210+)} < A_{(+)a}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(+)b} >< A_{(+)c}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(+)d} >$$

$$-\frac{1}{4608}\mathsf{g}^{(210)2}h_{ab}^{(210-)}h_{cd}^{(210-)} < A_{(-)a}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(-)b} >< A_{(-)c}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(-)d} >$$

$$-\frac{1}{2304}\mathsf{g}^{(210)2}h_{ab}^{(210+)}h_{cd}^{(210-)} < A_{(+)a}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(+)b} >< A_{(-)c}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(-)d} > \tag{28}$$

13

SU(5)expansion of these expressions gives

$$-\frac{1}{4608}\mathbf{g}^{(210)2}h_{ab}^{(210+)}h_{cd}^{(210+)} < A_{(+)a}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(+)b} >< A_{(+)c}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(+)d} >$$

$$= \mathbf{g}^{(210)2}\{-\frac{1}{6144}\left(\eta_{ab,cd}^{(210++)} + 8\eta_{ad,cb}^{(210++)}\right)\mathbf{A}_{(+)aij}^\dagger\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(+)ckl}^\dagger\mathbf{A}_{(+)d}^{kl}$$

$$-\frac{1}{768}\left(\eta_{ab,cd}^{(210++)} + 8\eta_{ad,cb}^{(210++)}\right)\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(+)c}^{i\dagger}\mathbf{A}_{(+)di}$$

$$+\frac{1}{512}\left(11\eta_{ab,cd}^{(210++)} - 2\eta_{ad,cb}^{(210++)}\right)\mathbf{A}_{(+)aij}^\dagger\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(+)c}^{k\dagger}\mathbf{A}_{(+)dk}$$

$$+\frac{1}{192}\left(\eta_{ab,cd}^{(210++)} + 2\eta_{ad,cb}^{(210++)}\right)\mathbf{A}_{(+)aij}^\dagger\mathbf{A}_{(+)b}^{jk}\mathbf{A}_{(+)c}^{i\dagger}\mathbf{A}_{(+)dk}$$

$$+\frac{1}{1536}\left(25\eta_{ab,cd}^{(210++)} + 18\eta_{ad,cb}^{(210++)}\right)\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(+)c}^{j\dagger}\mathbf{A}_{(+)dj}$$

$$+\frac{1}{1536}\left(\eta_{ab,cd}^{(210++)} - 6\eta_{ad,cb}^{(210++)}\right)\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(+)cij}^\dagger\mathbf{A}_{(+)d}^{ij}$$

$$+\eta_{ab,cd}^{(210++)}[\frac{1}{1536}\epsilon_{ijklm}\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(+)c}^{k\dagger}\mathbf{A}_{(+)d}^{lm} + \frac{1}{1536}\epsilon^{ijklm}\mathbf{A}_{(+)aij}^\dagger\mathbf{A}_{(+)bk}\mathbf{A}_{(+)clm}^\dagger\mathbf{A}_{(+)d}$$

$$+\frac{1}{768}\mathbf{A}_{(+)aij}^\dagger\mathbf{A}_{(+)b}^{jk}\mathbf{A}_{(+)ckl}^\dagger\mathbf{A}_{(+)d}^{li} - \frac{5}{1536}\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(+)c}^\dagger\mathbf{A}_{(+)d}]\}(29)$$

$$-\frac{1}{4608}\mathbf{g}^{(210)2}h_{ab}^{(210-)}h_{cd}^{(210-)} < A_{(-)a}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(-)b} >< A_{(-)c}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(-)d} >$$

$$= \mathbf{g}^{(210)2}\{-\frac{1}{6144}\left(\eta_{ab,cd}^{(210--)} + 16\eta_{ad,cb}^{(210--)}\right)\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)bij}\mathbf{A}_{(-)c}^{kl\dagger}\mathbf{A}_{(-)d}^{kl}$$

$$-\frac{1}{768}\left(11\eta_{ab,cd}^{(210--)} + 8\eta_{ad,cb}^{(210--)}\right)\mathbf{A}_{(-)a}^\dagger\mathbf{A}_{(-)b}\mathbf{A}_{(-)ci}^\dagger\mathbf{A}_{(-)d}^{i}$$

$$-\frac{1}{1536}\left(5\eta_{ab,cd}^{(210--)} + 6\eta_{ad,cb}^{(210--)}\right)\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)bij}\mathbf{A}_{(-)ck}^\dagger\mathbf{A}_{(-)d}^{k}$$

$$+\frac{1}{384}\left(\eta_{ab,cd}^{(210--)} + \eta_{ad,cb}^{(210--)}\right)\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)bjk}\mathbf{A}_{(-)ci}^\dagger\mathbf{A}_{(-)d}^{k}$$

$$+\frac{1}{1536}\left(\eta_{ab,cd}^{(210--)} - 6\eta_{ad,cb}^{(210--)}\right)\mathbf{A}_{(-)ai}^\dagger\mathbf{A}_{(-)b}^{i}\mathbf{A}_{(-)cj}^\dagger\mathbf{A}_{(-)d}^{j}$$

$$-\frac{1}{1536}\left(29\eta_{ab,cd}^{(210-)} + 18\eta_{ad,cb}^{(210--)}\right)\mathbf{A}_{(-)a}^\dagger\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^{ij\dagger}\mathbf{A}_{(-)dij}$$

$$+\eta_{ab,cd}^{(210--)}[\frac{5}{1536}\epsilon_{ijklm}\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^{kl\dagger}\mathbf{A}_{(-)d}^{m} + \frac{5}{1536}\epsilon^{ijklm}\mathbf{A}_{(-)ai}^\dagger\mathbf{A}_{(-)bjk}\mathbf{A}_{(-)c}^\dagger\mathbf{A}_{(-)dlm}$$

$$-\frac{1}{256}\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)bjk}\mathbf{A}_{(-)c}^{kl\dagger}\mathbf{A}_{(-)dli} + \frac{25}{1536}\mathbf{A}_{(-)a}^\dagger\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^\dagger\mathbf{A}_{(-)d}]\}(30)$$

$$-\frac{1}{2304}\mathbf{g}^{(210)2}h_{ab}^{(210+)}h_{cd}^{(210-)} < A_{(+)a}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(+)b} >< A_{(-)c}|\Gamma_{[\mu}\Gamma_\nu\Gamma_\rho\Gamma_{\lambda]}|A_{(-)d} >$$

$$= \frac{\mathbf{g}^{(210)2}\eta_{ab,cd}^{(210+-)}}{192}\{-10\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(-)c}^\dagger\mathbf{A}_{(-)d} + 2\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(-)c}^{ij\dagger}\mathbf{A}_{(-)dij}$$

$$-4\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(-)ci}^\dagger\mathbf{A}_{(-)d}^{i} - 32\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)bi}\mathbf{A}_{(-)c}^\dagger\mathbf{A}_{(-)d}^{i}$$

$$-4\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{\dagger}_{(-)c}\mathbf{A}_{(-)dij} - 32\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)b}\mathbf{A}^{\dagger}_{(-)ci}\mathbf{A}_{(-)d}$$

$$+12\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{\dagger}_{(-)ci}\mathbf{A}_{(-)djk} - 8\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{\dagger}_{(-)cj}\mathbf{A}_{(-)dki}$$

$$+18\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}\mathbf{A}^{jk\dagger}_{(-)c}\mathbf{A}_{(-)djk} + 4\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}\mathbf{A}^{\dagger}_{(-)cj}\mathbf{A}^{j}_{(-)d}$$

$$-56\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bj}\mathbf{A}^{jk\dagger}_{(-)c}\mathbf{A}_{(-)dki} - 24\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bj}\mathbf{A}^{\dagger}_{(-)ci}\mathbf{A}^{j}_{(-)d}$$

$$-164\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}\mathbf{A}^{\dagger}_{(-)c}\mathbf{A}_{(-)d} - 4\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)b}\mathbf{A}^{ij\dagger}_{(-)c}\mathbf{A}_{(-)d}$$

$$-8\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bk}\mathbf{A}^{jk\dagger}_{(-)c}\mathbf{A}^{i}_{(-)d} + 8\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bk}\mathbf{A}^{ij\dagger}_{(-)c}\mathbf{A}^{k}_{(-)d}$$

$$-\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{kl\dagger}_{(-)c}\mathbf{A}_{(-)dkl} + 8\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{il\dagger}_{(-)c}\mathbf{A}_{(-)dlk}$$

$$+8\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{\dagger}_{(-)ck}\mathbf{A}^{i}_{(-)d} - 38\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{\dagger}_{(-)c}\mathbf{A}_{(-)d}$$

$$+2\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{\dagger}_{(-)ck}\mathbf{A}^{k}_{(-)d} - 8\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{kl}_{(+)b}\mathbf{A}^{ij\dagger}_{(-)c}\mathbf{A}_{(-)dkl}$$

$$+14\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bk}\mathbf{A}^{\dagger}_{(-)c}\mathbf{A}_{(-)dlm} + 14\epsilon_{ijklm}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{lm\dagger}_{(-)c}\mathbf{A}_{(-)d}$$

$$-2\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)b}\mathbf{A}^{\dagger}_{(-)ck}\mathbf{A}_{(-)dlm} - 2\epsilon_{ijklm}\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{kl\dagger}_{(-)c}\mathbf{A}^{m}_{(-)d}\}(31)$$

where η's are defined by

$$\eta^{(210++)}_{ab,cd} = h^{(210+)}_{ab}h^{(210+)}_{cd}; \qquad \eta^{(210--)}_{ab,cd} = h^{(210-)}_{ab}h^{(210-)}_{cd}; \qquad \eta^{(210+-)}_{ab,cd} = h^{(210+)}_{ab}h^{(210-)}_{cd} \quad (32)$$

We also find

$$\mathsf{L}^{(210)}_{(1)auxiliary} + \mathsf{L}^{(210)}_{(2)auxiliary} + \mathsf{L}^{(210)}_{(6)auxiliary}$$

$$= -\left(\mu^*[h^{(210-)}]^{-1}[h^{(210-)}]^{\mathbf{T}}[h^{(210-)}]^{-1}\mu\right)_{ab}\left[\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}_{(+)b} + \frac{1}{4}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{ij}_{(+)b} + \mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}\right]$$

$$-\left(\mu[h^{(210+)}]^{-1\mathbf{T}}h^{(210+)}[h^{(210+)}]^{-1\mathbf{T}}\mu^*\right)_{ab}\left[\mathbf{A}^{\mathbf{T}}_{(-)a}\mathbf{A}^*_{(-)b} + \frac{1}{4}\mathbf{A}^{\mathbf{T}}_{(-)aij}\mathbf{A}^{ij*}_{(-)b} + \mathbf{A}^{i\mathbf{T}}_{(-)a}\mathbf{A}^*_{(-)bi}\right](33)$$

8 Appendix D

In this appendix we exhibit, for the benefit of the reader, a few SU(5) expansions of the terms appearing in our final Lagrangian Eq.(14). We begin by noting that any of the chiral fields $S (\equiv A, \Psi, F)$ can be expanded in terms of its SU(5) components as

$$|S_a>= |0> \mathbf{S}_a + \frac{1}{2}b^{\dagger}_i b^{\dagger}_j|0> \mathbf{S}^{ij}_a + \frac{1}{24}\epsilon^{ijklm}b^{\dagger}_j b^{\dagger}_k b^{\dagger}_l b^{\dagger}_m|0> \mathbf{S}_{ai} \qquad (34)$$

Together with the normalizations of appendix B and the basic theorem given in Ref.[3], we can expand terms such as

$$\frac{h^{(210)}_{ab}}{48m\sqrt{2}}\left(\overline{\Psi}_{aL}\gamma^A\widetilde{\Gamma}_{\mathsf{XY}}\Lambda_{L\mathsf{XY}}\right)\partial_A A_b = \frac{h^{(210)}_{ab}}{48m\sqrt{2}}\left(\overline{\Psi}_{aL}\gamma^A\Gamma_{[\mu}\Gamma_{\nu}\Gamma_{\rho}\Gamma_{\lambda]}\partial_A A_b\right)\Lambda_{L\mu\nu\rho\sigma}$$

$$= \frac{h^{(210)}_{ab}}{m}[\frac{1}{8\sqrt{30}}\left(10\overline{\Psi}_{aL}\gamma^A\partial_A\mathbf{A}_b - \overline{\Psi}_{aijL}\gamma^A\partial_A\mathbf{A}^{ij}_b + 2\overline{\Psi}^i_{aL}\gamma^A\partial_A\mathbf{A}_{bi}\right)\Lambda'_L$$

15

$$+\frac{1}{2\sqrt{3}}\left(\overline{\boldsymbol{\Psi}}^i_{aL}\gamma^A\partial_A\mathbf{A}_b\right)\Lambda^{'}_{iL}+\frac{1}{2\sqrt{3}}\left(\overline{\boldsymbol{\Psi}}_{aL}\gamma^A\partial_A\mathbf{A}_{bi}\right)\Lambda^{'i}_{L}$$

$$+\frac{1}{48}\left(-6\overline{\boldsymbol{\Psi}}^i_{almL}\gamma^A\partial_A\mathbf{A}_b+\epsilon_{ijklm}\overline{\boldsymbol{\Psi}}^i_{aL}\gamma^A\partial_A\mathbf{A}^{jk}_b\right)\Lambda^{'lm}_{L}$$

$$+\frac{1}{48}\left(-6\overline{\boldsymbol{\Psi}}_{aL}\gamma^A\partial_A\mathbf{A}^{lm}_b+\epsilon^{ijklm}\overline{\boldsymbol{\Psi}}_{aijL}\gamma^A\partial_A\mathbf{A}_{bk}\right)\Lambda^{'}_{lmL}$$

$$+\frac{1}{12}\left(-3\overline{\boldsymbol{\Psi}}^j_{aL}\gamma^A\partial_A\mathbf{A}^i_b+\overline{\boldsymbol{\Psi}}_{aikL}\gamma^A\partial_A\mathbf{A}^{kj}_b\right)\Lambda^{'i}_{jL}$$

$$+\frac{1}{12\sqrt{3}}\left(\epsilon_{ijklm}\overline{\boldsymbol{\Psi}}^i_{aL}\gamma^A\partial_A\mathbf{A}^{jn}_b\right)\Lambda^{'klm}_{nL}-\frac{1}{12\sqrt{3}}\left(\epsilon^{ijklm}\overline{\boldsymbol{\Psi}}_{ainL}\gamma^A\partial_A\mathbf{A}_{bj}\right)\Lambda^{'n}_{klmL}$$

$$-\frac{1}{8\sqrt{3}}\left(\overline{\boldsymbol{\Psi}}_{aijL}\gamma^A\partial_A\mathbf{A}^{kl}_b\right)\Lambda^{'ij}_{klL}]\quad(35)$$

$$B_{\texttt{WY}}\mathcal{V}^A_{\texttt{YZ}}\mathcal{V}_{A\texttt{ZW}}=B_{\mu\nu\rho\sigma}\mathcal{V}^A_{\rho\sigma\lambda\tau}\mathcal{V}_{A\lambda\tau\mu\nu}=B_{\mu\nu\rho\sigma}V_{\mu\nu\rho\sigma}$$

$$=\frac{1}{16}\{B_{c_ic_jc_kc_l}\,V_{\bar{c}_i\bar{c}_j\bar{c}_k\bar{c}_l}+B_{\bar{c}_i\bar{c}_j\bar{c}_k\bar{c}_l}\,V_{c_ic_jc_kc_l}+4B_{c_ic_jc_k\bar{c}_l}\,V_{\bar{c}_i\bar{c}_j\bar{c}_kc_l}$$

$$+4B_{\bar{c}_i\bar{c}_j\bar{c}_kc_l}\,V_{c_ic_jc_k\bar{c}_l}+6B_{c_ic_j\bar{c}_k\bar{c}_l}\,V_{\bar{c}_i\bar{c}_jc_kc_l}\}\quad(36)$$

where $V_{\mu\nu\rho\sigma}=\mathcal{V}^A_{\rho\sigma\lambda\tau}\mathcal{V}_{A\lambda\tau\mu\nu}$. Thus we find

$$B_{c_ic_jc_kc_l}\,V_{\bar{c}_i\bar{c}_j\bar{c}_k\bar{c}_l}=\frac{1}{2}B_{c_ic_jc_kc_l}\left(\mathcal{V}_{A\bar{c}_i\bar{c}_jc_mc_n}\mathcal{V}^A_{\bar{c}_m\bar{c}_n\bar{c}_k\bar{c}_l}+\mathcal{V}_{A\bar{c}_i\bar{c}_j\bar{c}_mc_n}\mathcal{V}^A_{c_m\bar{c}_n\bar{c}_k\bar{c}_l}\right)$$

$$=\frac{1}{144}B_i\mathcal{V}^j_A\mathcal{V}^{Ai}_j-\frac{1}{480}B_i\mathcal{V}^i_A\mathcal{V}^A-\frac{1}{48}\epsilon^{ijklm}B_i\mathcal{V}^p_{Ajkn}\mathcal{V}^{An}_{plm}$$

$$+\frac{1}{36}\epsilon^{ijklm}B_i\mathcal{V}^n_{Ajkl}\mathcal{V}^A_{nm}-\frac{1}{144}\epsilon^{ijklm}B_i\mathcal{V}_{Ajk}\mathcal{V}^A_{lm}\quad(37)$$

$$B_{\bar{c}_i\bar{c}_j\bar{c}_k\bar{c}_l}\,V_{c_ic_jc_kc_l}=\frac{1}{2}B_{\bar{c}_i\bar{c}_j\bar{c}_k\bar{c}_l}\left(\mathcal{V}_{Ac_ic_jc_mc_n}\mathcal{V}^A_{\bar{c}_m\bar{c}_nc_kc_l}+\mathcal{V}_{Ac_ic_j\bar{c}_mc_n}\mathcal{V}^A_{c_m\bar{c}_nc_kc_l}\right)$$

$$=\frac{1}{144}B^i\mathcal{V}_{Aj}\mathcal{V}^{Aj}_i-\frac{1}{480}B^i\mathcal{V}_{Ai}\mathcal{V}^A-\frac{1}{48}\epsilon_{ijklm}B^i\mathcal{V}^{jkn}_{Ap}\mathcal{V}^{Aplm}_n$$

$$+\frac{1}{36}\epsilon_{ijklm}B^i\mathcal{V}^{jkl}_{An}\mathcal{V}^{Anm}-\frac{1}{144}\epsilon_{ijklm}B^i\mathcal{V}^{ij}_A\mathcal{V}^{Akl}\quad(38)$$

$$4B_{c_ic_jc_k\bar{c}_l}\,V_{\bar{c}_i\bar{c}_j\bar{c}_kc_l}=B_{c_ic_jc_k\bar{c}_l}(\mathcal{V}_{A\bar{c}_i\bar{c}_jc_mc_n}\mathcal{V}^A_{\bar{c}_m\bar{c}_n\bar{c}_kc_l}+\mathcal{V}_{A\bar{c}_i\bar{c}_j\bar{c}_m\bar{c}_n}\mathcal{V}^A_{c_mc_n\bar{c}_kc_l}$$

$$+2\mathcal{V}_{A\bar{c}_i\bar{c}_j\bar{c}_mc_n}\mathcal{V}^A_{c_m\bar{c}_n\bar{c}_kc_l})$$

$$=B^{ijk}_l\mathcal{V}^{mn}_{Aij}\mathcal{V}^{Al}_{mnk}+2B^{ijk}_l\mathcal{V}^{lm}_{Akn}\mathcal{V}^{An}_{mij}-\frac{2}{3}B^{ijk}_l\mathcal{V}^{lm}_{Ajk}\mathcal{V}^A_{mi}$$

$$-\frac{2}{3}B^{ijk}_l\mathcal{V}^l_{Ajkm}\mathcal{V}^{Am}_i+\frac{2}{3}B^{ijk}_l\mathcal{V}^m_{Aijk}\mathcal{V}^{Al}_m+\frac{2}{9}B^{ijk}_l\mathcal{V}_{Ajk}\mathcal{V}^{Al}_i$$

$$-\frac{2}{3}B^{ij}\mathcal{V}^k_{Ajlm}\mathcal{V}^{Alm}_{ki}-\frac{2}{3}B^{ij}\mathcal{V}^k_{Aijl}\mathcal{V}^{Al}_k+\frac{7}{9}B^{ij}\mathcal{V}^{kl}_{Aij}\mathcal{V}^A_{kl}$$

$$+\frac{28}{27}B^{ij}\mathcal{V}_{Ajk}\mathcal{V}^{Ak}_i-\frac{1}{10}B^{ij}\mathcal{V}_{Aij}\mathcal{V}^A-\frac{1}{24}\epsilon_{ijklm}B^{ijn}_p\mathcal{V}^k_A\mathcal{V}^{Almp}_n$$

$$+\frac{1}{36}\epsilon_{ijklm}B^{ijk}_n\mathcal{V}^l_A\mathcal{V}^{Amn}-\frac{1}{36}\epsilon_{ijklm}B^{in}\mathcal{V}^j_A\mathcal{V}^{Aklm}_n+\frac{1}{36}\epsilon_{ijklm}B^{ij}\mathcal{V}^k_A\mathcal{V}^{Alm}\quad(39)$$

$$4B_{\bar{c}_i\bar{c}_j\bar{c}_k c_l} V_{c_i c_j c_k \bar{c}_l} = B_{\bar{c}_i\bar{c}_j\bar{c}_k c_l}(\mathcal{V}_{Ac_i c_j c_m c_n}\mathcal{V}^A_{\bar{c}_m\bar{c}_n c_k \bar{c}_l} + \mathcal{V}_{Ac_i c_j \bar{c}_m \bar{c}_n}\mathcal{V}^A_{c_m c_n c_k \bar{c}_l}$$
$$+2\mathcal{V}_{Ac_i c_j \bar{c}_m c_n}\mathcal{V}^A_{c_m \bar{c}_n c_k \bar{c}_l})$$
$$= B^l_{ijk}\mathcal{V}^{Aij}_{mn}\mathcal{V}^{mnk}_{Al} + 2B^l_{ijk}\mathcal{V}^{Akn}_{lm}\mathcal{V}^{mij}_{An} - \frac{2}{3}B^l_{ijk}\mathcal{V}^{Ajk}_{lm}\mathcal{V}^{mi}_A$$
$$-\frac{2}{3}B^l_{ijk}\mathcal{V}^{Ajkm}_l\mathcal{V}^i_{Am} + \frac{2}{3}B^l_{ijk}\mathcal{V}^{Aijk}_m\mathcal{V}^m_{Al} + \frac{2}{9}B^l_{ijk}\mathcal{V}^{Ajk}\mathcal{V}^i_{Al}$$
$$-\frac{2}{3}B_{ij}\mathcal{V}^{Ajlm}_k\mathcal{V}^{ki}_{Alm} - \frac{2}{3}B_{ij}\mathcal{V}^{Aijl}_k\mathcal{V}^k_{Al} + \frac{7}{9}B_{ij}\mathcal{V}^{Aij}_{kl}\mathcal{V}^{kl}_A$$
$$+\frac{28}{27}B_{ij}\mathcal{V}^{Ajk}\mathcal{V}^i_{Ak} - \frac{1}{10}B_{ij}\mathcal{V}^{Aij}\mathcal{V}_A - \frac{1}{24}\epsilon^{ijklm}B^p_{ijn}\mathcal{V}^A_k\mathcal{V}^n_{Almp}$$
$$+\frac{1}{36}\epsilon^{ijklm}B^n_{ijk}\mathcal{V}^A_l\mathcal{V}_{Amn} - \frac{1}{36}\epsilon^{ijklm}B_{in}\mathcal{V}^A_j\mathcal{V}^n_{Aklm} + \frac{1}{36}\epsilon^{ijklm}B_{ij}\mathcal{V}^A_k\mathcal{V}_{Alm} \tag{40}$$

$$6B_{c_i c_j \bar{c}_k \bar{c}_l} V_{\bar{c}_i\bar{c}_j c_k c_l} = B_{c_i c_j \bar{c}_k \bar{c}_l}(\frac{3}{2}\mathcal{V}_{A\bar{c}_i\bar{c}_j c_m c_n}\mathcal{V}^A_{\bar{c}_m\bar{c}_n c_k c_l} + \frac{3}{2}\mathcal{V}_{A\bar{c}_i\bar{c}_j \bar{c}_m \bar{c}_n}\mathcal{V}^A_{c_m c_n c_k c_l}$$
$$+3\mathcal{V}_{A\bar{c}_i\bar{c}_j \bar{c}_m c_n}\mathcal{V}^A_{c_m \bar{c}_n c_k c_l})$$
$$= \frac{3}{2}B^{ij}_{kl}\mathcal{V}^{Amn}_{ij}\mathcal{V}^{kl}_{Amn} + 2B^{ij}_{kl}\mathcal{V}^{Aln}_{ij}\mathcal{V}^k_{An} - \frac{3}{20}B^{ij}_{kl}\mathcal{V}^{Akl}_{ij}\mathcal{V}_A$$
$$+2B^{ij}_{kl}\mathcal{V}^{Akl}_{jm}\mathcal{V}^m_{Ai} + \frac{4}{3}B^{ij}_{kl}\mathcal{V}^{Al}_j\mathcal{V}^k_{Ai} + 2B^i_j\mathcal{V}^{Akl}_{im}\mathcal{V}^{mj}_{Akl}$$
$$+\frac{4}{3}B^i_j\mathcal{V}^{Ajk}_{il}\mathcal{V}^l_{Ak} - \frac{4}{9}B^i_j\mathcal{V}^{Aj}_k\mathcal{V}^k_{Ai} - \frac{2}{15}B^i_j\mathcal{V}^{Aj}_i\mathcal{V}_A$$
$$-\frac{3}{20}B\mathcal{V}^{Aij}_{kl}\mathcal{V}^{kl}_{Aij} - \frac{2}{15}B\mathcal{V}^{Ai}_j\mathcal{V}^j_{Ai} - \frac{3}{200}B\mathcal{V}^A\mathcal{V}_A$$
$$+\frac{1}{48}B^i_j\mathcal{V}^{Aj}\mathcal{V}_{Ai} - \frac{1}{480}B\mathcal{V}^{Ai}\mathcal{V}_{Ai} + 3B^{ij}_{kl}\mathcal{V}^{An}_{ijm}\mathcal{V}^{mkl}_{An}$$
$$+2B^{ij}_{kl}\mathcal{V}^{Al}_{ijm}\mathcal{V}^{mk}_A + 2B^{ij}_{kl}\mathcal{V}^{Aklm}_j\mathcal{V}_{Ami} + \frac{1}{3}B^{ij}_{kl}\mathcal{V}^A_{ij}\mathcal{V}^{kl}_A$$
$$-4B^i_m\mathcal{V}^{Aj}_{ikl}\mathcal{V}^{klm}_{Aj} - \frac{4}{3}B^i_j\mathcal{V}^{Aj}_{ikl}\mathcal{V}^{kl}_A - \frac{4}{3}B^i_j\mathcal{V}^{Ajkl}_i\mathcal{V}_{Akl}$$
$$+2B^i_k\mathcal{V}^A_{ij}\mathcal{V}^{jk}_A - \frac{3}{10}B\mathcal{V}^{Ai}_{jkl}\mathcal{V}^{jkl}_{Ai} - \frac{1}{6}B\mathcal{V}^A_{ij}\mathcal{V}^{ij}_A \tag{41}$$

9 Appendix E

In this appendix we give the complete supersymmetric couplings containing the singlet of SO(10).

$$\mathsf{L} = \mathsf{L}^{(1\ K.E.)}_V + \mathsf{L}^{(1\ Interaction)}_{V+\Phi} + \mathsf{L}^{(1)}_\mathsf{W} \tag{42}$$

where

$$\mathsf{L}^{(1\ K.E.)}_V = \frac{1}{64}\left[\widehat{\mathcal{W}}^{\tilde{\alpha}}\widehat{\mathcal{W}}_{\tilde{\alpha}}|_{\theta^2} + \widehat{\overline{\mathcal{W}}}_{\dot{\tilde{\alpha}}}\widehat{\overline{\mathcal{W}}}^{\dot{\tilde{\alpha}}}|_{\bar{\theta}^2}\right] \tag{43}$$

$$\widehat{\mathcal{W}}^{\tilde{\alpha}} = \overline{\mathsf{D}}^2\mathsf{D}_{\tilde{\alpha}}\widehat{\mathsf{V}} \tag{44}$$

Thus we have

$$
\mathsf{L}_V^{(1\ K.E.)} = -\frac{1}{4}\mathcal{V}_{AB}\mathcal{V}^{AB} - \frac{i}{2}\overline{\Lambda}\gamma^A \mathcal{D}_A \Lambda + \mathsf{L}_{(1)auxiliary}^{(1)}
$$

$$
\mathcal{V}^{AB} = \partial^A \mathcal{V}_{\mu\nu}^B - \partial^B \mathcal{V}_{\mu\nu}^A
$$

$$
\mathsf{L}_{(1)auxiliary}^{(1)} = \frac{1}{2}D^2
$$

$$
\Lambda = \begin{pmatrix} \lambda_{\tilde{\alpha}} \\ \overline{\lambda}^{\dot{\tilde{\alpha}}} \end{pmatrix} \tag{45}
$$

Further,

$$
\mathsf{L}_{V+\Phi}^{(1\ Interaction)} = h_{ab}^{(1+)} < \widehat{\Phi}_{(+)a}|e^{\mathsf{g}^{(1)}q^{(+)}\widehat{\mathsf{V}}}|\widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2}
$$

$$
+ h_{ab}^{(1-)} < \widehat{\Phi}_{(-)a}|e^{\mathsf{g}^{(1)}q^{(-)}\widehat{\mathsf{V}}}|\widehat{\Phi}_{(-)b} > |_{\theta^2\bar{\theta}^2} \tag{46}
$$

where $q^{(\pm)}$ are the U(1) charges. Expanding $e^{\mathsf{g}^{(1)}q^{(\pm)}}$ we have

$$
\mathsf{L}_{V+\Phi}^{(1\ Interaction)} = h_{ab}^{(1+)}[< \widehat{\Phi}_{(+)a}|\widehat{\Phi}_{(+)b} > + \mathsf{g}^{(1)}q^{(+)} < \widehat{\Phi}_{(+)a}|\widehat{\mathsf{V}}|\widehat{\Phi}_{(+)b} >
$$

$$
+ \frac{1}{2}\mathsf{g}^{(1)2}q^{(+)2} < \widehat{\Phi}_{(+)a}|\widehat{\mathsf{V}}^2|\widehat{\Phi}_{(+)b} >]|_{\theta^2\bar{\theta}^2} + h_{ab}^{(1-)}[< \widehat{\Phi}_{(-)a}|\widehat{\Phi}_{(-)b} >
$$

$$
+ \mathsf{g}^{(1)}q^{(-)} < \widehat{\Phi}_{(-)a}|\widehat{\mathsf{V}}|\widehat{\Phi}_{(-)b} > + \frac{1}{2}\mathsf{g}^{(1)2}q^{(-)2} < \widehat{\Phi}_{(-)a}|\widehat{\mathsf{V}}^2|\widehat{\Phi}_{(-)b} >]|_{\theta^2\bar{\theta}^2} \tag{47}
$$

where the quantities entering Eq.(47) are determined by Eqs.(48) -(54) below

$$
h_{ab}^{(1+)} < \widehat{\Phi}_{(+)a}|\widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2} = h_{ab}^{(1+)}[-\partial_A \mathbf{A}_{(+)a}^{\dagger}\partial^A \mathbf{A}_{(+)b} - \partial_A \mathbf{A}_{(+)a}^{i\dagger}\partial^A \mathbf{A}_{(+)bi}
$$

$$
-\partial_A \mathbf{A}_{(+)aij}^{\dagger}\partial^A \mathbf{A}_{(+)b}^{ij} - i\overline{\mathbf{\Psi}}_{(+)aL}\gamma^A\partial_A \mathbf{\Psi}_{(+)bL}
$$

$$
-i\overline{\mathbf{\Psi}}_{(+)aL}^i\gamma^A\partial_A \mathbf{\Psi}_{(+)biL} - i\overline{\mathbf{\Psi}}_{(+)aijL}\gamma^A\partial_A \mathbf{\Psi}_{(+)bL}^{ij}] + \mathsf{L}_{(2)auxiliary}^{(1)}
$$

$$
\mathsf{L}_{(2)auxiliary}^{(1)} = h_{ab}^{(1+)} < F_{(+)a}|F_{(+)b} > \tag{48}
$$

$$
h_{ab}^{(1-)} < \widehat{\Phi}_{(-)a}|\widehat{\Phi}_{(-)b} > |_{\theta^2\bar{\theta}^2} = h_{ab}^{(1-)}[-\partial_A \mathbf{A}_{(-)a}^{\dagger}\partial^A \mathbf{A}_{(-)b} - \partial_A \mathbf{A}_{(-)ai}^{\dagger}\partial^A \mathbf{A}_{(-)b}^i
$$

$$
-\partial_A \mathbf{A}_{(-)a}^{ij\dagger}\partial^A \mathbf{A}_{(-)bij} - i\overline{\mathbf{\Psi}}_{(-)aL}\gamma^A\partial_A \mathbf{\Psi}_{(-)bL}
$$

$$
-i\overline{\mathbf{\Psi}}_{(-)aiL}\gamma^A\partial_A \mathbf{\Psi}_{(-)bL}^i - i\overline{\mathbf{\Psi}}_{(-)aL}^{ij}\gamma^A\partial_A \mathbf{\Psi}_{(-)bijL}] + \mathsf{L}_{(3)auxiliary}^{(1)}
$$

$$
\mathsf{L}_{(3)auxiliary}^{(1)} = h_{ab}^{(1-)} < F_{(-)a}|F_{(-)b} > \tag{49}
$$

$$
\mathbf{\Psi}_{(\pm)a} = \begin{pmatrix} \psi_{(\pm)a\tilde{\alpha}} \\ \overline{\psi}_{(\pm)a}^{\dot{\tilde{\alpha}}} \end{pmatrix} \tag{50}
$$

$$
h_{ab}^{(1+)}\mathsf{g}^{(1)}q^{(+)} < \widehat{\Phi}_{(+)a}|\widehat{\mathsf{V}}|\widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2}
$$

18

$$= h_{ab}^{(1+)} \mathsf{g}^{(1)} q^{(+)} \{ [\frac{1}{2} \left(i \mathbf{A}_{(+)a}^{\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b} - \overline{\mathbf{\Psi}}_{(+)aL} \gamma_A \mathbf{\Psi}_{(+)bL} \right)$$

$$+ \frac{1}{4} \left(i \mathbf{A}_{(+)aij}^{\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{ij} - \overline{\mathbf{\Psi}}_{(+)aijL} \gamma_A \mathbf{\Psi}_{(+)bL}^{ij} \right)$$

$$+ \frac{1}{2} \left(i \mathbf{A}_{(+)a}^{i\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)bi} - \overline{\mathbf{\Psi}}_{(+)aL}^{i} \gamma_A \mathbf{\Psi}_{(+)biL} \right)] \mathcal{V}^A$$

$$+ \frac{i}{\sqrt{2}} \left[\mathbf{A}_{(+)a}^{\dagger} \overline{\mathbf{\Psi}}_{(+)bR} + \frac{1}{2} \mathbf{A}_{(+)aij}^{\dagger} \overline{\mathbf{\Psi}}_{(+)bR}^{ij} + \mathbf{A}_{(+)a}^{i\dagger} \overline{\mathbf{\Psi}}_{(+)biR} \right] \Lambda_L$$

$$- \frac{i}{\sqrt{2}} \left[\overline{\mathbf{\Psi}}_{(+)aL} \mathbf{A}_{(+)b} + \frac{1}{12} \overline{\mathbf{\Psi}}_{(+)aijL} \mathbf{A}_{(+)b}^{ij} + \overline{\mathbf{\Psi}}_{(+)aL}^{i} \mathbf{A}_{(+)bi} \right] \Lambda_R \} + \mathsf{L}_{(4)auxiliary}^{(1)}$$

$$\mathsf{L}_{(4)auxiliary}^{(1)} = \frac{h_{ab}^{(1+)} \mathsf{g}^{(1)} q^{(+)}}{2} < A_{(+)a} | A_{(+)b} > D \quad (51)$$

$$h_{ab}^{(1-)} \mathsf{g}^{(1)} q^{(-)} < \widehat{\Phi}_{(-)a} | \widehat{\mathsf{V}} | \widehat{\Phi}_{(-)b} > |_{\theta^2 \bar{\theta}^2}$$

$$= h_{ab}^{(1-)} \mathsf{g}^{(1)} q^{(-)} \{ [\frac{1}{2} \left(i \mathbf{A}_{(-)a}^{\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(-)b} - \overline{\mathbf{\Psi}}_{(-)aL} \gamma_A \mathbf{\Psi}_{(-)bL} \right)$$

$$+ \frac{1}{4} \left(\mathbf{A}_{(-)a}^{ij\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(-)bij} + i \overline{\mathbf{\Psi}}_{(-)aL}^{ij} \gamma_A \mathbf{\Psi}_{(-)bijL} \right)$$

$$+ \frac{1}{2} \left(\mathbf{A}_{(-)ai}^{\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(-)b}^{i} + i \overline{\mathbf{\Psi}}_{(-)aiL} \gamma_A \mathbf{\Psi}_{(-)bL}^{i} \right)] \mathcal{V}^A$$

$$+ \frac{i}{\sqrt{2}} \left[\mathbf{A}_{(-)a}^{\dagger} \overline{\mathbf{\Psi}}_{(-)bR} + \frac{1}{2} \mathbf{A}_{(-)a}^{ij\dagger} \overline{\mathbf{\Psi}}_{(-)bijR} + \mathbf{A}_{(-)ai}^{\dagger} \overline{\mathbf{\Psi}}_{(-)bR}^{i} \right] \Lambda_L$$

$$- \frac{i}{\sqrt{2}} \left[\overline{\mathbf{\Psi}}_{(-)aL} \mathbf{A}_{(-)b} + \frac{1}{2} \overline{\mathbf{\Psi}}_{(-)aL}^{ij} \mathbf{A}_{(-)bij} + \overline{\mathbf{\Psi}}_{(-)aiL} \mathbf{A}_{(-)b}^{i} \right] \Lambda_R \} + \mathsf{L}_{(5)auxiliary}^{(1)}$$

$$\mathsf{L}_{(5)auxiliary}^{(1)} = \frac{h_{ab}^{(1-)} \mathsf{g}^{(1)} q^{(-)}}{2} < A_{(-)a} | A_{(-)b} > D \quad (52)$$

$$\frac{1}{2} h_{ab}^{(1+)} \mathsf{g}^{(1)2} q^{(+)2} < \widehat{\Phi}_{(+)a} | \widehat{\mathsf{V}}^2 | \widehat{\Phi}_{(+)b} > |_{\theta^2 \bar{\theta}^2}$$

$$= \frac{h_{ab}^{(1+)} \mathsf{g}^{(1)2} q^{(+)2}}{4} [\mathbf{A}_{(+)a}^{\dagger} \mathbf{A}_{(+)b} + \frac{1}{2} \mathbf{A}_{(+)aij}^{\dagger} \mathbf{A}_{(+)b}^{ij} + \mathbf{A}_{(+)a}^{i\dagger} \mathbf{A}_{(+)bi}] \mathcal{V}_A \mathcal{V}^A \quad (53)$$

$$\frac{1}{2} h_{ab}^{(1-)} \mathsf{g}^{(1)2} q^{(-)2} < \widehat{\Phi}_{(-)a} | \widehat{\mathsf{V}}^2 | \widehat{\Phi}_{(-)b} > |_{\theta^2 \bar{\theta}^2}$$

$$= \frac{h_{ab}^{(1-)} \mathsf{g}^{(1)2} q^{(-)2}}{4} [\mathbf{A}_{(-)a}^{\dagger} \mathbf{A}_{(-)b} + \frac{1}{2} \mathbf{A}_{(-)a}^{ij\dagger} \mathbf{A}_{(-)bij} + \mathbf{A}_{(-)ai}^{\dagger} \mathbf{A}_{(-)b}^{i}] \mathcal{V}_A \mathcal{V}^A \quad (54)$$

Finally, $\mathsf{L}_{\mathsf{W}}^{(1)}$ appearing in Eq.(42) is given by

$$\mathsf{L}_{\mathsf{W}}^{(1)} = \mu_{ab} < \widehat{\Phi}_{(-)a}^{*} | B | \widehat{\Phi}_{(+)b} > |_{\theta^2} + h.c. \quad (55)$$

19

Evaluation of Eq.(55) gives

$$\mathsf{L}_W^{(1)} = -i\mu_{ab}\left(\overline{\mathbf{\Psi}}_{(-)aR}\mathbf{\Psi}_{(+)bL} + \overline{\mathbf{\Psi}}_{(-)aR}^i\mathbf{\Psi}_{(+)biL} - \frac{1}{2}\overline{\mathbf{\Psi}}_{(-)aijR}\mathbf{\Psi}_{(+)bL}^{ij}\right)$$

$$+i\mu_{ab}^*\left(\overline{\mathbf{\Psi}}_{(-)aL}\mathbf{\Psi}_{(+)bR} + \overline{\mathbf{\Psi}}_{(-)aiL}\mathbf{\Psi}_{(+)bR}^i - \frac{1}{2}\overline{\mathbf{\Psi}}_{(-)aL}^{ij}\mathbf{\Psi}_{(+)bijR}\right) + \mathsf{L}_{(6)auxiliary}^{(1)}$$

$$\mathsf{L}_{(6)auxiliary}^{(1)} = i\mu_{ab}[\mathbf{F}_{(-)a}\mathbf{A}_{(+)b} + \mathbf{A}_{(-)a}^{\mathbf{T}}\mathbf{F}_{(+)b} - \frac{1}{2}\mathbf{F}_{(-)aij}\mathbf{A}_{(+)b}^{ij}$$

$$-\frac{1}{2}\mathbf{A}_{(-)aij}^{\mathbf{T}}\mathbf{F}_{(+)b}^{ij} + \mathbf{F}_{(-)a}^i\mathbf{A}_{(+)bi} + \mathbf{A}_{(-)a}^{i\mathbf{T}}\mathbf{F}_{(+)bi}] + h.c. \quad (56)$$

Elimination of the auxiliary fields $\mathbf{F}_{(\pm)}$ through their field equations gives

$$\mathsf{L}_{(2)auxiliary}^{(1)} + \mathsf{L}_{(3)auxiliary}^{(1)} + \mathsf{L}_{(6)auxiliary}^{(1)}$$

$$= -\left(\mu^*[h^{(1-)}]^{-1}[h^{(1-)}]^{\mathbf{T}}[h^{(1-)}]^{-1}\mu\right)_{ab}\left[\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b} + \frac{1}{4}\mathbf{A}_{(+)aij}^\dagger\mathbf{A}_{(+)b}^{ij} + \mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\right]$$

$$-\left(\mu[h^{(1+)}]^{-\mathbf{1T}}h^{(1+)}[h^{(1+)}]^{-\mathbf{1T}}\mu^*\right)_{ab}\left[\mathbf{A}_{(-)a}^{\mathbf{T}}\mathbf{A}_{(-)b}^* + \frac{1}{4}\mathbf{A}_{(-)aij}^{\mathbf{T}}\mathbf{A}_{(-)b}^{ij*} + \mathbf{A}_{(-)a}^{i\mathbf{T}}\mathbf{A}_{(-)bi}^*\right](57)$$

Similarly, after eliminating the field D we get

$$\mathsf{L}_{(1)auxiliary}^{(1)} + \mathsf{L}_{(4)auxiliary}^{(1)} + \mathsf{L}_{(5)auxiliary}^{(1)}$$

$$= -\frac{1}{8}\mathsf{g}^{(1)2}h_{ab}^{(1+)}h_{cd}^{(1+)}q^{(+)2} < A_{(+)a}|A_{(+)b} >< A_{(+)c}|A_{(+)d} >$$

$$-\frac{1}{8}\mathsf{g}^{(1)2}h_{ab}^{(1-)}h_{cd}^{(1-)}q^{(-)2} < A_{(-)a}|A_{(-)b} >< A_{(-)c}|A_{(-)d} >$$

$$-\frac{1}{4}\mathsf{g}^{(45)2}h_{ab}^{(1+)}h_{cd}^{(1-)}q^{(+)}q^{(-)} < A_{(+)a}|A_{(+)b} >< A_{(-)c}|A_{(-)d} > \quad (58)$$

$$-\frac{1}{8}\mathsf{g}^{(1)2}h_{ab}^{(1+)}h_{cd}^{(1+)}q^{(+)2} < A_{(+)a}|A_{(+)b} >< A_{(+)c}|A_{(+)d} >$$

$$= -\frac{1}{8}\mathsf{g}^{(1)2}h_{ab}^{(1+)}h_{cd}^{(1+)}q^{(+)2}[\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(+)c}^\dagger\mathbf{A}_{(+)d} + \frac{1}{4}\mathbf{A}_{(+)aij}^\dagger\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(+)ckl}^\dagger\mathbf{A}_{(+)d}^{kl}$$

$$+\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(+)c}^{j\dagger}\mathbf{A}_{(+)dj} + \mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(+)cij}^\dagger\mathbf{A}_{(+)d}^{ij}$$

$$+2\mathbf{A}_{(+)a}^\dagger\mathbf{A}_{(+)b}\mathbf{A}_{(+)c}^{i\dagger}\mathbf{A}_{(+)di} + \mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(+)cjk}^\dagger\mathbf{A}_{(+)d}^{jk}] \quad (59)$$

$$-\frac{1}{8}\mathsf{g}^{(1)2}h_{ab}^{(1-)}h_{cd}^{(1-)}q^{(-)2} < A_{(-)a}|A_{(-)b} >< A_{(-)c}|A_{(-)d} >$$

$$= -\frac{1}{8}\mathsf{g}^{(1)2}h_{ab}^{(1-)}h_{cd}^{(1-)}q^{(-)2}[\mathbf{A}_{(-)a}^\dagger\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^\dagger\mathbf{A}_{(-)d} + \frac{1}{4}\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)bij}\mathbf{A}_{(-)c}^{kl\dagger}\mathbf{A}_{(-)dkl}$$

$$+\mathbf{A}_{(-)ai}^\dagger\mathbf{A}_{(-)b}^i\mathbf{A}_{(-)cj}^\dagger\mathbf{A}_{(-)d}^j + \mathbf{A}_{(-)a}^\dagger\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^{ij\dagger}\mathbf{A}_{(-)dij}$$

$$+2\mathbf{A}_{(-)a}^\dagger\mathbf{A}_{(-)b}\mathbf{A}_{(-)ci}^\dagger\mathbf{A}_{(-)d}^i + \mathbf{A}_{(-)ai}^\dagger\mathbf{A}_{(-)b}^i\mathbf{A}_{(-)c}^{jk\dagger}\mathbf{A}_{(-)djk}] \quad (60)$$

$$-\frac{1}{4}\mathsf{g}^{(1)2}h^{(1+)}_{ab}h^{(1-)}_{cd}q^{(+)}q^{(-)} < A_{(+)a}|A_{(+)b} >< A_{(-)c}|A_{(-)d} >$$

$$= -\frac{1}{4}\mathsf{g}^{(1)2}h^{(1-)}_{ab}h^{(1-)}_{cd}q^{(+)}q^{(-)}[\mathbf{A}^\dagger_{(+)a}\mathbf{A}_{(+)b}\mathbf{A}^\dagger_{(-)c}\mathbf{A}_{(-)d} + \frac{1}{2}\mathbf{A}^\dagger_{(+)a}\mathbf{A}_{(+)b}\mathbf{A}^{ij\dagger}_{(-)c}\mathbf{A}_{(-)dij}$$

$$+\mathbf{A}^\dagger_{(+)a}\mathbf{A}_{(+)b}\mathbf{A}^\dagger_{(-)ci}\mathbf{A}^i_{(-)d} + \frac{1}{2}\mathbf{A}^\dagger_{(+)aij}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^\dagger_{(-)c}\mathbf{A}_{(-)d}$$

$$+\frac{1}{4}\mathbf{A}^\dagger_{(+)aij}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{kl\dagger}_{(-)c}\mathbf{A}_{(-)dkl} + \frac{1}{2}\mathbf{A}^\dagger_{(+)aij}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^\dagger_{(-)ck}\mathbf{A}^k_{(-)d}$$

$$+\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}\mathbf{A}^\dagger_{(-)c}\mathbf{A}_{(-)d} + \frac{1}{2}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}\mathbf{A}^{kl\dagger}_{(-)c}\mathbf{A}_{(-)dkl} + \mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}\mathbf{A}^\dagger_{(-)cj}\mathbf{A}^j_{(-)d}](61)$$

10   Appendix F

In this appendix we give the complete supersymmetric vector couplings for the 45-dimensional tensor of SO(10) in the Wess-Zumino gauge

$$\mathsf{L}^{45} = \mathsf{L}_V^{(45\ K.E.)} + \mathsf{L}_{V+\Phi}^{(45\ Interaction)} + \mathsf{L}_\mathsf{W}^{(45)} \tag{62}$$

where

$$\mathsf{L}_V^{(45\ K.E.)} = \frac{1}{64}\left[\widehat{\mathcal{W}}^{\tilde{\alpha}}\widehat{\mathcal{W}}_{\tilde{\alpha}}|_{\theta^2} + \widehat{\overline{\mathcal{W}}}_{\dot{\tilde{\alpha}}}\widehat{\overline{\mathcal{W}}}^{\dot{\tilde{\alpha}}}|_{\bar{\theta}^2}\right]$$

$$\widehat{\mathcal{W}}_{\tilde{\alpha}} = \frac{1}{\mathsf{g}^{(45)}}\overline{\mathsf{D}}^2 e^{-\mathsf{g}^{(45)}\widehat{\mathsf{V}}_{\mu\nu}M_{\mu\nu}}\mathsf{D}_{\tilde{\alpha}}e^{\mathsf{g}^{(45)}\widehat{\mathsf{V}}_{\rho\lambda}M_{\rho\lambda}} \tag{63}$$

where $M_{\mu\nu}$ are the 45 generators in the vector (10-dimensional) representation that satisfy the following Lie algebra

$$[M_{\alpha\beta}, M_{\gamma\rho}] = -i\left(\delta_{\beta\gamma}M_{\alpha\rho} + \delta_{\alpha\rho}M_{\beta\gamma} - \delta_{\alpha\gamma}M_{\beta\rho} - \delta_{\beta\rho}M_{\alpha\gamma}\right) \tag{64}$$

and take on the form

$$(M_{\mu\nu})_{\alpha\beta} = -i\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) \tag{65}$$

and $\widehat{\mathsf{V}}_{\mu\nu}$ in the Wess-Zumino gauge is given by

$$\widehat{\mathsf{V}}_{\mu\nu} = -\theta\sigma^A\bar{\theta}\mathcal{V}_{A\mu\nu} + i\theta^2\bar{\theta}\overline{\lambda}_{\mu\nu} - i\bar{\theta}^2\theta\lambda_{\mu\nu} + \frac{1}{2}\theta^2\bar{\theta}^2 D_{\mu\nu} \tag{66}$$

Finally we have

$$\mathsf{L}_V^{(45\ K.E.)} = -\frac{1}{4}\mathcal{V}_{AB\mu\nu}\mathcal{V}_{\mu\nu}^{AB} - \frac{i}{2}\overline{\Lambda}_{\mu\nu}\gamma^A\mathcal{D}_A\Lambda_{\mu\nu} + \mathsf{L}_{(1)auxiliary}^{(45)}$$

$$\mathcal{V}_{\mu\nu}^{AB} = \partial^A\mathcal{V}_{\mu\nu}^B - \partial^B\mathcal{V}_{\mu\nu}^A + \mathsf{g}^{(45)}\left(\mathcal{V}_{\mu\alpha}^A\mathcal{V}_{\alpha\nu}^B - \mathcal{V}_{\mu\alpha}^B\mathcal{V}_{\alpha\nu}^A\right)$$

$$\mathcal{D}^A = \partial^A + \frac{i\mathbf{g}^{(45)}}{2} M_{\mu\nu} \mathcal{V}^A_{\mu\nu}$$

$$\mathcal{D}^A \Lambda_{\mu\nu} = \partial^A \Lambda_{\mu\nu} + \mathbf{g}^{(45)} \left(\mathcal{V}^A_{\mu\alpha} \Lambda_{\alpha\nu} - \mathcal{V}^A_{\nu\alpha} \Lambda_{\nu\alpha} \right)$$

$$\mathsf{L}^{(45)}_{(1)auxiliary} = \frac{1}{2} D_{\mu\nu} D_{\mu\nu}$$

$$\Lambda_{\mu\nu} = \begin{pmatrix} \lambda_{\tilde{\alpha}\mu\nu} \\ \overline{\lambda}^{\dot{\tilde{\alpha}}}_{\mu\nu} \end{pmatrix} \tag{67}$$

To normalize the SU(5) fields appearing in $-\frac{1}{4}\mathcal{V}_{AB\mu\nu}\mathcal{V}^{AB}_{\mu\nu}$, we carry out a field redefinition

$$\mathcal{V}_A = 2\sqrt{5}\mathcal{V}'_A; \qquad \mathcal{V}^j_{Ai} = \sqrt{2}\mathcal{V}'^{j}_{Ai}; \qquad \mathcal{V}^{ij}_A = \sqrt{2}\mathcal{V}'^{ij}_A; \qquad \mathcal{V}_{Aij} = \sqrt{2}\mathcal{V}'_{Aij}$$

$$\Lambda = \sqrt{10}\Lambda'; \qquad \Lambda^j_i = \sqrt{2}\Lambda'^{j}_i; \qquad \Lambda^{ij} = \sqrt{2}\Lambda'^{ij}; \qquad \Lambda_{ij} = \sqrt{2}\Lambda'_{ij} \tag{68}$$

so that

$$-\frac{1}{4}\mathcal{V}^{AB}_{\mu\nu}\mathcal{V}_{AB\mu\nu} = -\frac{1}{2}\mathcal{V}'_{AB}\mathcal{V}'^{AB\dagger} - \frac{1}{2!}\frac{1}{2}\mathcal{V}'^{ij}_{AB}\mathcal{V}'^{ABij\dagger} - \frac{1}{4}\mathcal{V}'^{i}_{ABj}\mathcal{V}'^{ABj}_i$$

$$-\frac{i}{2}\overline{\Lambda}_{\mu\nu}\gamma^A \mathcal{D}_A \Lambda_{\mu\nu} = -\frac{i}{2}\frac{1}{2!}\overline{\Lambda}'_{ij}\gamma^A \mathcal{D}_A \Lambda'_{ij} - \frac{i}{2}\frac{1}{2!}\overline{\Lambda}'^{ij}\gamma^A \mathcal{D}_A \Lambda'^{ij}$$

$$-\frac{i}{2}\overline{\Lambda}'^{i}_j \gamma^A \mathcal{D}_A \Lambda'^{i}_j - \frac{i}{2}\overline{\Lambda}' \gamma^A \mathcal{D}_A \Lambda' \tag{69}$$

Next we look at the second term in Eq.(62)

$$\mathsf{L}^{(45\ Interaction)}_{V+\Phi} = h^{(45+)}_{ab} < \widehat{\Phi}_{(+)a} | e^{\frac{1}{2!}\mathbf{g}^{(45)}\widehat{V}_{\mu\nu}\Sigma_{\mu\nu}} | \widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2}$$

$$+ h^{(45-)}_{ab} < \widehat{\Phi}_{(-)a} | e^{\frac{1}{2!}\mathbf{g}^{(45)}\widehat{V}_{\mu\nu}\Sigma_{\mu\nu}} | \widehat{\Phi}_{(-)b} > |_{\theta^2\bar{\theta}^2} \tag{70}$$

$\Sigma_{\mu\nu}$ being the 45 generators in the spinorial representation. We find

$$\mathsf{L}^{(45\ Interaction)}_{V+\Phi} = h^{(45+)}_{ab}[< \widehat{\Phi}_{(+)a} | \widehat{\Phi}_{(+)b} > + \frac{1}{2}\mathbf{g}^{(45)} < \widehat{\Phi}_{(+)a} | \widehat{V}_{\mu\nu}\Sigma_{\mu\nu} | \widehat{\Phi}_{(+)b} >$$

$$+ \frac{1}{8}\mathbf{g}^{(45)2} < \widehat{\Phi}_{(+)a} | \widehat{V}_{\mu\nu}\Sigma_{\mu\nu}\widehat{V}_{\rho\lambda}\Sigma_{\rho\lambda} | \widehat{\Phi}_{(+)b} >]|_{\theta^2\bar{\theta}^2} + h^{(45-)}_{ab}[< \widehat{\Phi}_{(-)a} | \widehat{\Phi}_{(-)b} >$$

$$+ \frac{1}{2}\mathbf{g}^{(45)} < \widehat{\Phi}_{(-)a} | \widehat{V}_{\mu\nu}\Sigma_{\mu\nu} | \widehat{\Phi}_{(-)b} > + \frac{1}{8}\mathbf{g}^{(45)2} < \widehat{\Phi}_{(-)a} | \widehat{V}_{\mu\nu}\Sigma_{\mu\nu}\widehat{V}_{\rho\lambda}\Sigma_{\rho\lambda} | \widehat{\Phi}_{(-)b} >]|_{\theta^2\bar{\theta}^2} \tag{71}$$

where

$$h^{(45+)}_{ab} < \widehat{\Phi}_{(+)a} | \widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2} = h^{(45+)}_{ab}[-\partial_A \mathbf{A}^{\dagger}_{(+)a}\partial^A \mathbf{A}_{(+)b} - \partial_A \mathbf{A}^{i\dagger}_{(+)a}\partial^A \mathbf{A}_{(+)bi}$$

$$-\partial_A \mathbf{A}^{\dagger}_{(+)aij}\partial^A \mathbf{A}^{ij}_{(+)b} - i\overline{\boldsymbol{\Psi}}_{(+)aL}\gamma^A \partial_A \boldsymbol{\Psi}_{(+)bL}$$

$$-i\overline{\boldsymbol{\Psi}}^{i}_{(+)aL}\gamma^A \partial_A \boldsymbol{\Psi}_{(+)biL} - i\overline{\boldsymbol{\Psi}}_{(+)aijL}\gamma^A \partial_A \boldsymbol{\Psi}^{ij}_{(+)bL}] + \mathsf{L}^{(45)}_{(2)auxiliary}$$

$$\mathsf{L}^{(45)}_{(2)auxiliary} = h^{(45+)}_{ab} < F_{(+)a} | F_{(+)b} > \tag{72}$$

22

$$h_{ab}^{(45-)} < \widehat{\Phi}_{(-)a}|\widehat{\Phi}_{(-)b} > |_{\theta^2\bar{\theta}^2} = h_{ab}^{(45-)}[-\partial_A \mathbf{A}_{(-)a}^\dagger \partial^A \mathbf{A}_{(-)b} - \partial_A \mathbf{A}_{(-)ai}^\dagger \partial^A \mathbf{A}_{(-)b}^i$$

$$-\partial_A \mathbf{A}_{(-)a}^{ij\dagger} \partial^A \mathbf{A}_{(-)bij} - i\overline{\mathbf{\Psi}}_{(-)aL}\gamma^A \partial_A \mathbf{\Psi}_{(-)bL}$$

$$-i\overline{\mathbf{\Psi}}_{(-)aiL}\gamma^A \partial_A \mathbf{\Psi}_{(-)bL}^i - i\overline{\mathbf{\Psi}}_{(-)aL}^{ij}\gamma^A \partial_A \mathbf{\Psi}_{(-)bijL}] + \mathsf{L}_{(3)auxiliary}^{(45)}$$

$$\mathsf{L}_{(3)auxiliary}^{(45)} = h_{ab}^{(45-)} < F_{(-)a}|F_{(-)b} > \quad (73)$$

$$\mathbf{\Psi}_{(\pm)a} = \begin{pmatrix} \psi_{(\pm)a\tilde{\alpha}} \\ \overline{\psi}_{(\pm)a}^{\dot{\alpha}} \end{pmatrix} \quad (74)$$

$$\frac{1}{2}h_{ab}^{(45+)}\mathsf{g}^{(45)} < \widehat{\Phi}_{(+)a}|\widehat{\mathsf{V}}_{\mu\nu}\Sigma_{\mu\nu}|\widehat{\Phi}_{(+)b} > |_{\theta^2\bar{\theta}^2}$$

$$= h_{ab}^{(45+)}\mathsf{g}^{(45)}\{[-\frac{\sqrt{5}}{2}\left(\mathbf{A}_{(+)a}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b} + i\overline{\mathbf{\Psi}}_{(+)aL}\gamma_A\mathbf{\Psi}_{(+)bL}\right)$$

$$-\frac{1}{4\sqrt{5}}\left(\mathbf{A}_{(+)aij}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{ij} + i\overline{\mathbf{\Psi}}_{(+)aijL}\gamma_A\mathbf{\Psi}_{(+)bL}^{ij}\right)$$

$$+\frac{3}{2\sqrt{5}}\left(\mathbf{A}_{(+)a}^{i\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)bi} + i\overline{\mathbf{\Psi}}_{(+)aL}^i\gamma_A\mathbf{\Psi}_{(+)biL}\right)]\mathcal{V}'^A$$

$$+[\frac{1}{2\sqrt{2}}\left(\mathbf{A}_{(+)alm}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b} + i\overline{\mathbf{\Psi}}_{(+)almL}\gamma_A\mathbf{\Psi}_{(+)bL}\right)$$

$$+\frac{1}{4\sqrt{2}}\epsilon_{ijklm}\left(\mathbf{A}_{(+)a}^{i\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{jk} + i\overline{\mathbf{\Psi}}_{(+)aL}^i\gamma_A\mathbf{\Psi}_{(+)bL}^{jk}\right)]\mathcal{V}'^{Alm}$$

$$+[-\frac{1}{2\sqrt{2}}\left(\mathbf{A}_{(+)a}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{lm} + i\overline{\mathbf{\Psi}}_{(+)aL}\gamma_A\mathbf{\Psi}_{(+)bL}^{lm}\right)$$

$$-\frac{1}{4\sqrt{2}}\epsilon^{ijklm}\left(\mathbf{A}_{(+)aij}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)bk} + i\overline{\mathbf{\Psi}}_{(+)aijL}\gamma_A\mathbf{\Psi}_{(+)bkL}\right)]\mathcal{V}'^A_{lm}$$

$$+[-\frac{1}{\sqrt{2}}\left(\mathbf{A}_{(+)aik}^\dagger \overleftrightarrow{\partial}_A \mathbf{A}_{(+)b}^{kj} + i\overline{\mathbf{\Psi}}_{(+)aikL}\gamma_A\mathbf{\Psi}_{(+)bL}^{kj}\right)$$

$$-\frac{1}{\sqrt{2}}\left(\mathbf{A}_{(+)a}^{j\dagger} \overleftrightarrow{\partial}_A \mathbf{A}_{(+)bi} + i\overline{\mathbf{\Psi}}_{(+)aL}^j\gamma_A\mathbf{\Psi}_{(+)biL}\right)]\mathcal{V}_j'^{iA}$$

$$+\frac{\sqrt{5}}{2}\left[-\mathbf{A}_{(+)a}^\dagger \overline{\mathbf{\Psi}}_{(+)bR} - \frac{1}{10}\mathbf{A}_{(+)aij}^\dagger \overline{\mathbf{\Psi}}_{(+)bR}^{ij} + \frac{3}{5}\mathbf{A}_{(+)a}^{i\dagger} \overline{\mathbf{\Psi}}_{(+)biR}\right]\Lambda_L'$$

$$+\frac{1}{2}\left[\mathbf{A}_{(+)alm}^\dagger \overline{\mathbf{\Psi}}_{(+)bR} + \frac{1}{2}\epsilon_{ijklm}\mathbf{A}_{(+)a}^{i\dagger} \overline{\mathbf{\Psi}}_{(+)bR}^{jk}\right]\Lambda_L'^{lm}$$

$$-\frac{1}{2}\left[\mathbf{A}_{(+)a}^\dagger \overline{\mathbf{\Psi}}_{(+)bR}^{lm} + \frac{1}{2}\epsilon^{ijklm}\mathbf{A}_{(+)aij}^\dagger \overline{\mathbf{\Psi}}_{(+)bkR}\right]\Lambda_{lmL}'$$

$$-\left[\mathbf{A}_{(+)aik}^\dagger \overline{\mathbf{\Psi}}_{(+)bR}^{kj} + \mathbf{A}_{(+)a}^{j\dagger} \overline{\mathbf{\Psi}}_{(+)biR}\right]\Lambda_{jL}'^i$$

$$-\frac{\sqrt{5}}{2}\left[-\overline{\mathbf{\Psi}}_{(+)aL}\mathbf{A}_{(+)b} - \frac{1}{10}\overline{\mathbf{\Psi}}_{(+)aijL}\mathbf{A}_{(+)b}^{ij} + \frac{3}{5}\overline{\mathbf{\Psi}}_{(+)aL}^i\mathbf{A}_{(+)bi}\right]\Lambda_R'$$

$$-\frac{1}{2}\left[\overline{\mathbf{\Psi}}_{(+)almL}\mathbf{A}_{(+)b} + \frac{1}{2}\epsilon_{ijklm}\overline{\mathbf{\Psi}}_{(+)aL}^i\mathbf{A}_{(+)b}^{jk}\right]\Lambda_R'^{lm}$$

$$+\frac{1}{2}\left[\overline{\mathbf{\Psi}}_{(+)aL}\mathbf{A}_{(+)b}^{lm} + \frac{1}{2}\epsilon^{ijklm}\overline{\mathbf{\Psi}}_{(+)aijL}\mathbf{A}_{(+)bk}\right]\Lambda_{lmR}'$$

$$+ \left[\overline{\boldsymbol{\Psi}}_{(+)aikL} \mathbf{A}_{(+)b}^{kj} + \overline{\boldsymbol{\Psi}}_{(+)aL}^{j} \mathbf{A}_{(+)bi} \right] \Lambda_{jR}^{'i} \} + \mathsf{L}_{(4)auxiliary}^{(45)}$$

$$\mathsf{L}_{(4)auxiliary}^{(45)} = \frac{h_{ab}^{(45+)} \mathsf{g}^{(45)}}{4} < A_{(+)a} | \Sigma_{\mu\nu} | A_{(+)b} > D_{\mu\nu} \qquad (75)$$

where

$$\mathbf{A}_{(+)a}^{i\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(+)bi} \overset{def}{=} \mathbf{A}_{(+)a}^{i\dagger} \partial_A \mathbf{A}_{(+)bi} - \left(\partial_A \mathbf{A}_{(+)a}^{i\dagger} \right) \mathbf{A}_{(+)bi} \qquad (76)$$

Similarly for $\overline{16}_- 16_- -$ couplings we have

$$\frac{1}{2} h_{ab}^{(45-)} \mathsf{g}^{(45)} < \widehat{\Phi}_{(-)a} | \widehat{\mathsf{V}}_{\mu\nu} \Sigma_{\mu\nu} | \widehat{\Phi}_{(-)b} > |_{\theta^2 \bar{\theta}^2}$$

$$= h_{ab}^{(45-)} \mathsf{g}^{(45)} \{ [\frac{\sqrt{5}}{2} \left(\mathbf{A}_{(-)a}^{\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)b} + i \overline{\boldsymbol{\Psi}}_{(-)aL} \gamma_A \boldsymbol{\Psi}_{(-)bL} \right)$$

$$+ \frac{1}{4\sqrt{5}} \left(\mathbf{A}_{(-)a}^{ij\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)bij} + i \overline{\boldsymbol{\Psi}}_{(-)aL}^{ij} \gamma_A \boldsymbol{\Psi}_{(-)bijL} \right)$$

$$- \frac{3}{2\sqrt{5}} \left(\mathbf{A}_{(-)ai}^{\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)b}^{i} + i \overline{\boldsymbol{\Psi}}_{(-)aiL} \gamma_A \boldsymbol{\Psi}_{(-)bL}^{i} \right)] \mathcal{V}'^A$$

$$+ [\frac{1}{2\sqrt{2}} \left(\mathbf{A}_{(-)a}^{\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)blm} + i \overline{\boldsymbol{\Psi}}_{(-)aL} \gamma_A \boldsymbol{\Psi}_{(-)blmL} \right)$$

$$+ \frac{1}{4\sqrt{2}} \epsilon_{ijklm} \left(\mathbf{A}_{(-)a}^{ij\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)b}^{k} + i \overline{\boldsymbol{\Psi}}_{(-)aL}^{ij} \gamma_A \boldsymbol{\Psi}_{(-)bL}^{k} \right)] \mathcal{V}'^{Alm}$$

$$+ [-\frac{1}{2\sqrt{2}} \left(\mathbf{A}_{(-)a}^{lm\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)b} + i \overline{\boldsymbol{\Psi}}_{(-)aL}^{lm} \gamma_A \boldsymbol{\Psi}_{(-)bL} \right)$$

$$- \frac{1}{4\sqrt{2}} \epsilon^{ijklm} \left(\mathbf{A}_{(-)ai}^{\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)bjk} + i \overline{\boldsymbol{\Psi}}_{(-)aiL} \gamma_A \boldsymbol{\Psi}_{(-)bjkL} \right)] \mathcal{V}_{lm}'^A$$

$$+ [\frac{1}{\sqrt{2}} \left(\mathbf{A}_{(-)a}^{jk\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)bki} + i \overline{\boldsymbol{\Psi}}_{(-)aL}^{jk} \gamma_A \boldsymbol{\Psi}_{(-)bkiL} \right)$$

$$+ \frac{1}{\sqrt{2}} \left(\mathbf{A}_{(-)ai}^{\dagger} \overset{\leftrightarrow}{\partial}_A \mathbf{A}_{(-)b}^{j} + i \overline{\boldsymbol{\Psi}}_{(-)aiL} \gamma_A \boldsymbol{\Psi}_{(-)bL}^{j} \right)] \mathcal{V}_j'^{iA}$$

$$- \frac{\sqrt{5}}{2} \left[-\mathbf{A}_{(-)a}^{\dagger} \overline{\boldsymbol{\Psi}}_{(-)bR} - \frac{1}{10} \mathbf{A}_{(-)a}^{ij\dagger} \overline{\boldsymbol{\Psi}}_{(-)bijR} + \frac{3}{5} \mathbf{A}_{(-)ai}^{\dagger} \overline{\boldsymbol{\Psi}}_{(-)bR}^{i} \right] \Lambda_L'$$

$$+ \frac{1}{2} \left[\mathbf{A}_{(-)a}^{\dagger} \overline{\boldsymbol{\Psi}}_{(-)blmR} + \frac{1}{2} \epsilon_{ijklm} \mathbf{A}_{(-)a}^{ij\dagger} \overline{\boldsymbol{\Psi}}_{(-)bR}^{k} \right] \Lambda_L'^{lm}$$

$$- \frac{1}{2} \left[\mathbf{A}_{(-)a}^{lm\dagger} \overline{\boldsymbol{\Psi}}_{(-)bR} + \frac{1}{2} \epsilon^{ijklm} \mathbf{A}_{(-)ai}^{\dagger} \overline{\boldsymbol{\Psi}}_{(-)bjkR} \right] \Lambda_{lmL}'$$

$$+ \left[\mathbf{A}_{(-)a}^{jk\dagger} \overline{\boldsymbol{\Psi}}_{(-)bkiR} + \mathbf{A}_{(-)ai}^{\dagger} \overline{\boldsymbol{\Psi}}_{(-)bR}^{j} \right] \Lambda_{jL}'^{i}$$

$$+ \frac{\sqrt{5}}{2} \left[-\overline{\boldsymbol{\Psi}}_{(-)aL} \mathbf{A}_{(-)b} - \frac{1}{10} \overline{\boldsymbol{\Psi}}_{(-)aL}^{ij} \mathbf{A}_{(-)bij} + \frac{3}{5} \overline{\boldsymbol{\Psi}}_{(-)aiL} \mathbf{A}_{(-)b}^{i} \right] \Lambda_R'$$

$$- \frac{1}{2} \left[\overline{\boldsymbol{\Psi}}_{(-)aL} \mathbf{A}_{(-)blm} + \frac{1}{2} \epsilon_{ijklm} \overline{\boldsymbol{\Psi}}_{(-)aL}^{ij} \mathbf{A}_{(-)b}^{k} \right] \Lambda_R'^{lm}$$

$$+ \frac{1}{2} \left[\overline{\boldsymbol{\Psi}}_{(-)aL}^{lm} \mathbf{A}_{(-)b} + \frac{1}{2} \epsilon^{ijklm} \overline{\boldsymbol{\Psi}}_{(-)aiL} \mathbf{A}_{(-)bjk} \right] \Lambda_{lmR}'$$

$$- \left[\overline{\boldsymbol{\Psi}}_{(-)aL}^{jk} \mathbf{A}_{(-)bki} + \overline{\boldsymbol{\Psi}}_{(-)aiL} \mathbf{A}_{(-)b}^{j} \right] \Lambda_{jR}'^{i} \} + \mathsf{L}_{(5)auxiliary}^{(45)}$$

24

$$\mathsf{L}^{(45)}_{(5)auxiliary} = \frac{h_{ab}^{(45-)}\mathsf{g}^{(45)}}{4} < A_{(-)a}|\Sigma_{\mu\nu}|A_{(-)b} > D_{\mu\nu} \tag{77}$$

Next we evaluate couplings to $\overline{16}_+ 16_+$ of matter which are quadratic in the vector multiplet fields. We have

$$\frac{1}{8}h_{ab}^{(45+)}\mathsf{g}^{(45)2} < \widehat{\Phi}_{(+)a}|\widehat{\mathsf{V}}_{\mu\nu}\Sigma_{\mu\nu}\widehat{\mathsf{V}}_{\rho\lambda}\Sigma_{\rho\lambda}|\widehat{\Phi}_{(+)b} > |_{\theta^2\bar\theta^2}$$

$$= h_{ab}^{(45+)}\mathsf{g}^{(45)2}\{[-\frac{5}{4}[\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}_{(+)b} + \frac{1}{50}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{ij}_{(+)b} + \frac{9}{25}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)bi}]\mathcal{V}'_A\mathcal{V}'^A$$

$$+\frac{1}{2}[-\mathbf{A}^{m\dagger}_{(+)a}\mathbf{A}_{(+)bm}\delta^l_i\delta^k_j + \left(\mathbf{A}^{\dagger}_{(+)aim}\mathbf{A}^{ml}_{(+)b} - \mathbf{A}^{l\dagger}_{(+)a}\mathbf{A}_{(+)bi}\right)\delta^k_j + \mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{kl}_{(+)b}]\mathcal{V}'^i_{Ak}\mathcal{V}'^{Aj}_l$$

$$+\frac{1}{4}[\left(2\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}^{km}_{(+)b} + \epsilon^{ijpkm}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bp}\right)\delta^l_n - 2\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)ain}\mathbf{A}_{(+)bj}]\mathcal{V}'_{Akl}\mathcal{V}'^{An}_m$$

$$+\frac{1}{4}[\left(-2\mathbf{A}^{\dagger}_{(+)akm}\mathbf{A}_{(+)b} - \epsilon_{ijpkm}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jp}_{(+)b}\right)\delta^n_l + 2\epsilon_{ijklm}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{nj}_{(+)b}]\mathcal{V}'^{kl}_A\mathcal{V}'^{Am}_n$$

$$+\frac{1}{4\sqrt{10}}[-6\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}^{lm}_{(+)b} + \epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bk}]\mathcal{V}'_{Alm}\mathcal{V}'^A$$

$$+\frac{1}{4\sqrt{10}}[6\mathbf{A}^{\dagger}_{(+)alm}\mathbf{A}_{(+)b} - \epsilon_{ijklm}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jk}_{(+)b}]\mathcal{V}'^{lm}_A\mathcal{V}'^A$$

$$+\frac{1}{\sqrt{10}}[3\mathbf{A}^{j\dagger}_{(+)a}\mathbf{A}_{(+)bi} + 5\mathbf{A}^{\dagger}_{(+)aik}\mathbf{A}^{kj}_{(+)b}]\mathcal{V}'^i_{Aj}\mathcal{V}'^A$$

$$-\frac{1}{8}[\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}_{(+)bi}]\mathcal{V}'_{Ajk}\mathcal{V}'^A_{lm} - \frac{1}{8}[\epsilon_{ijklm}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}_{(+)b}]\mathcal{V}'^{jk}_A\mathcal{V}'^{Alm}$$

$$+\frac{1}{8}[\left(2\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}_{(+)b} + \mathbf{A}^{\dagger}_{(+)amn}\mathbf{A}^{mn}_{(+)b}\right)\delta^k_i\delta^l_j + 2\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}^{kl}_{(+)b}$$

$$+2\left(\mathbf{A}^{\dagger}_{(+)aim}\mathbf{A}^{ml}_{(+)b} - \mathbf{A}^{l\dagger}_{(+)a}\mathbf{A}_{(+)bi}\right)\delta^k_j]\mathcal{V}'^{ij}_A\mathcal{V}'^A_{kl}\}(78)$$

Similarly couplings to $\overline{16}_- 16_-$ of matter which are quadratic in the vector multiplet fields are given by

$$\frac{1}{8}h_{ab}^{(45-)}\mathsf{g}^{(45)2} < \widehat{\Phi}_{(-)a}|\widehat{\mathsf{V}}_{\mu\nu}\Sigma_{\mu\nu}\widehat{\mathsf{V}}_{\rho\lambda}\Sigma_{\rho\lambda}|\widehat{\Phi}_{(-)b} > |_{\theta^2\bar\theta^2}$$

$$= h_{ab}^{(45-)}\mathsf{g}^{(45)2}\{[-\frac{5}{4}[\mathbf{A}^{\dagger}_{(-)a}\mathbf{A}_{(-)b} + \frac{1}{50}\mathbf{A}^{ij\dagger}_{(-)a}\mathbf{A}_{(-)bij} + \frac{9}{25}\mathbf{A}^{\dagger}_{(-)ai}\mathbf{A}^i_{(-)b}]\mathcal{V}'_A\mathcal{V}'^A$$

$$+\frac{1}{2}[\left(\mathbf{A}^{im\dagger}_{(-)a}\mathbf{A}_{(-)bml} - \mathbf{A}^{\dagger}_{(-)al}\mathbf{A}^i_{(-)b}\right)\delta^j_k - \mathbf{A}^{ij\dagger}_{(-)a}\mathbf{A}_{(-)bkl}]\mathcal{V}'^k_{Ai}\mathcal{V}'^{Al}_j$$

$$+\frac{1}{4}[\left(2\mathbf{A}^{km\dagger}_{(-)a}\mathbf{A}_{(-)b} + \epsilon^{ijpkm}\mathbf{A}^{\dagger}_{(-)ai}\mathbf{A}_{(-)bjp}\right)\delta^l_n + \epsilon^{ijklm}\mathbf{A}^{\dagger}_{(-)an}\mathbf{A}_{(-)bij}]\mathcal{V}'_{Akl}\mathcal{V}'^{An}_m$$

$$-\frac{1}{4}[\left(2\mathbf{A}^{\dagger}_{(-)a}\mathbf{A}_{(-)bkm} + \epsilon_{ijpkm}\mathbf{A}^{ij\dagger}_{(-)a}\mathbf{A}^p_{(-)b}\right)\delta^n_l + \epsilon_{ijklm}\mathbf{A}^{ij\dagger}_{(-)a}\mathbf{A}^n_{(-)b}]\mathcal{V}'^{kl}_A\mathcal{V}'^{Am}_n$$

$$+\frac{1}{4\sqrt{10}}[6\mathbf{A}^{lm\dagger}_{(-)a}\mathbf{A}_{(-)b} - \epsilon^{ijklm}\mathbf{A}^{\dagger}_{(-)ai}\mathbf{A}_{(-)bjk}]\mathcal{V}'_{Alm}\mathcal{V}'^A$$

$$+\frac{1}{4\sqrt{10}}[-6\mathbf{A}^{\dagger}_{(-)a}\mathbf{A}_{(-)blm} + \epsilon_{ijklm}\mathbf{A}^{ij\dagger}_{(-)a}\mathbf{A}^k_{(-)b}]\mathcal{V}'^{lm}_A\mathcal{V}'^A$$

25

$$+\frac{1}{\sqrt{10}}[3\mathbf{A}_{(-)ai}^{\dagger}\mathbf{A}_{(-)b}^{j}+\mathbf{A}_{(-)a}^{jk\dagger}\mathbf{A}_{(-)bki}]\mathcal{V}_{Aj}^{\prime i}\mathcal{V}^{\prime A}$$

$$-\frac{1}{8}[\epsilon^{ijklm}\mathbf{A}_{(-)ai}^{\dagger}\mathbf{A}_{(-)b}]\mathcal{V}_{Ajk}^{\prime}\mathcal{V}_{lm}^{\prime A}-\frac{1}{8}[\epsilon_{ijklm}\mathbf{A}_{(-)a}^{\dagger}\mathbf{A}_{(-)b}^{i}]\mathcal{V}_{A}^{\prime jk}\mathcal{V}^{\prime Alm}$$

$$+\frac{1}{4}[\left(\mathbf{A}_{(-)a}^{\dagger}\mathbf{A}_{(-)b}+\mathbf{A}_{(-)a}^{mn\dagger}\mathbf{A}_{(-)bmn}+\mathbf{A}_{(-)am}^{\dagger}\mathbf{A}_{(-)b}^{m}\right)\delta_k^i\delta_l^j+\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)bkl}$$

$$+\left(-3\mathbf{A}_{(-)a}^{im\dagger}\mathbf{A}_{(-)bml}+\mathbf{A}_{(-)al}^{\dagger}\mathbf{A}_{(-)b}^{i}\right)\delta_k^j]\mathcal{V}_{Aij}^{\prime}\mathcal{V}^{\prime Akl}\}\quad(79)$$

$$\mathsf{L}_{\mathsf{W}}^{(45)}=\mathsf{W}(\widehat{\Phi}_{(+)},\widehat{\Phi}_{(-)})|_{\theta^2}+h.c.\quad(80)$$

where

$$\mathsf{W}(\widehat{\Phi}_{(+)},\widehat{\Phi}_{(-)})=\mu_{ab}<\widehat{\Phi}_{(-)a}^{*}|B|\widehat{\Phi}_{(+)b}>$$

$$\mathsf{W}(\mathbf{A}_{(+)},\mathbf{A}_{(-)})=i\mu_{ab}\left(\mathbf{A}_{(-)a}^{\mathbf{T}}\mathbf{A}_{(+)b}-\frac{1}{2}\mathbf{A}_{(-)aij}^{\mathbf{T}}\mathbf{A}_{(+)b}^{ij}+\mathbf{A}_{(-)a}^{i\mathbf{T}}\mathbf{A}_{(+)bi}\right)\quad(81)$$

and where μ_{ab} is taken to be a symmetric tensor. Thus we have

$$\mathsf{L}_{\mathsf{W}}^{(45)}=-i\mu_{ab}\left(\overline{\boldsymbol{\Psi}}_{(-)aR}\boldsymbol{\Psi}_{(+)bL}+\overline{\boldsymbol{\Psi}}_{(-)aR}^{i}\boldsymbol{\Psi}_{(+)biL}-\frac{1}{2}\overline{\boldsymbol{\Psi}}_{(-)aijR}\boldsymbol{\Psi}_{(+)bL}^{ij}\right)$$

$$+i\mu_{ab}^{*}\left(\overline{\boldsymbol{\Psi}}_{(-)aL}\boldsymbol{\Psi}_{(+)bR}+\overline{\boldsymbol{\Psi}}_{(-)aiL}\boldsymbol{\Psi}_{(+)bR}^{i}-\frac{1}{2}\overline{\boldsymbol{\Psi}}_{(-)aL}^{ij}\boldsymbol{\Psi}_{(+)bijR}\right)+\mathsf{L}_{(6)auxiliary}^{(45)}$$

$$\mathsf{L}_{(6)auxiliary}^{(45)}=i\mu_{ab}[\mathbf{F}_{(-)a}\mathbf{A}_{(+)b}+\mathbf{A}_{(-)a}^{\mathbf{T}}\mathbf{F}_{(+)b}-\frac{1}{2}\mathbf{F}_{(-)aij}\mathbf{A}_{(+)b}^{ij}$$

$$-\frac{1}{2}\mathbf{A}_{(-)aij}^{\mathbf{T}}\mathbf{F}_{(+)b}^{ij}+\mathbf{F}_{(-)a}^{i}\mathbf{A}_{(+)bi}+\mathbf{A}_{(-)a}^{i\mathbf{T}}\mathbf{F}_{(+)bi}]+h.c.\quad(82)$$

Eliminating the fields, $\mathbf{F}_{(\pm)}$ through their field equations we get

$$\mathsf{L}_{(2)auxiliary}^{(45)}+\mathsf{L}_{(3)auxiliary}^{(45)}+\mathsf{L}_{(6)auxiliary}^{(45)}$$

$$=-\left(\mu^{*}[h^{(45-)}]^{-\mathbf{1}}[h^{(45-)}]^{\mathbf{T}}[h^{(45-)}]^{-\mathbf{1}}\mu\right)_{ab}\left[\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}+\frac{1}{4}\mathbf{A}_{(+)aij}^{\dagger}\mathbf{A}_{(+)b}^{ij}+\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\right]$$

$$-\left(\mu[h^{(45+)}]^{-\mathbf{1T}}h^{(45+)}[h^{(45+)}]^{-\mathbf{1T}}\mu^{*}\right)_{ab}\left[\mathbf{A}_{(-)a}^{\mathbf{T}}\mathbf{A}_{(-)b}^{*}+\frac{1}{4}\mathbf{A}_{(-)aij}^{\mathbf{T}}\mathbf{A}_{(-)b}^{ij*}+\mathbf{A}_{(-)a}^{i\mathbf{T}}\mathbf{A}_{(-)bi}^{*}\right](83)$$

Similarly, eliminating the auxiliary field $D_{\mu\nu}$ we get

$$\mathsf{L}_{(1)auxiliary}^{(45)}+\mathsf{L}_{(4)auxiliary}^{(45)}+\mathsf{L}_{(5)auxiliary}^{(45)}$$

$$=-\frac{1}{32}\mathsf{g}^{(45)2}h_{ab}^{(45+)}h_{cd}^{(45+)}<A_{(+)a}|\Sigma_{\mu\nu}|A_{(+)b}><A_{(+)c}|\Sigma_{\mu\nu}|A_{(+)d}>$$

$$-\frac{1}{32}\mathsf{g}^{(45)2}h_{ab}^{(45-)}h_{cd}^{(45-)}<A_{(-)a}|\Sigma_{\mu\nu}|A_{(-)b}><A_{(-)c}|\Sigma_{\mu\nu}|A_{(-)d}>$$

$$-\frac{1}{16}\mathsf{g}^{(45)2}h_{ab}^{(45+)}h_{cd}^{(45-)}<A_{(+)a}|\Sigma_{\mu\nu}|A_{(+)b}><A_{(-)c}|\Sigma_{\mu\nu}|A_{(-)d}>\quad(84)$$

The terms above when expanded in terms of SU(5) fields give

$$-\frac{1}{32}\mathbf{g}^{(45)2}h_{ab}^{(45+)}h_{cd}^{(45+)} < A_{(+)a}|\Sigma_{\mu\nu}|A_{(+)b} >< A_{(+)c}|\Sigma_{\mu\nu}|A_{(+)d} >$$

$$= \mathbf{g}^{(45)2}\{-\frac{1}{16}\left(\eta_{ab,cd}^{(45++)} + 4\eta_{ad,cb}^{(45++)}\right)(\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}\mathbf{A}_{(+)cij}^{\dagger}\mathbf{A}_{(+)d}^{ij} + \mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(+)c}^{j\dagger}\mathbf{A}_{(+)dj}$$

$$+\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(+)cjk}^{\dagger}\mathbf{A}_{(+)d}^{jk}) - \frac{1}{2}\left(\eta_{ab,cd}^{(45++)} + \eta_{ad,cb}^{(45++)}\right)\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bj}\mathbf{A}_{(+)cik}^{\dagger}\mathbf{A}_{(+)d}^{kj}$$

$$+\eta_{ab,cd}^{(45++)}[-\frac{1}{8}\epsilon_{ijklm}\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(+)c}^{k\dagger}\mathbf{A}_{(+)d}^{lm} - \frac{1}{8}\epsilon^{ijklm}\mathbf{A}_{(+)aij}^{\dagger}\mathbf{A}_{(+)bk}\mathbf{A}_{(+)clm}^{\dagger}\mathbf{A}_{(+)d}$$

$$-\frac{1}{4}\mathbf{A}_{(+)aik}^{\dagger}\mathbf{A}_{(+)b}^{kj}\mathbf{A}_{(+)cjl}^{\dagger}\mathbf{A}_{(+)d}^{li} + \frac{3}{64}\mathbf{A}_{(+)aij}^{\dagger}\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(+)ckl}^{\dagger}\mathbf{A}_{(+)d}^{kl}$$

$$+\frac{3}{8}\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}\mathbf{A}_{(+)c}^{i\dagger}\mathbf{A}_{(+)di} - \frac{5}{16}\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}\mathbf{A}_{(+)c}^{\dagger}\mathbf{A}_{(+)d}]\}(85)$$

$$-\frac{1}{32}\mathbf{g}^{(45)2}h_{ab}^{(45-)}h_{cd}^{(45-)} < A_{(-)a}|\Sigma_{\mu\nu}|A_{(-)b} >< A_{(-)c}|\Sigma_{\mu\nu}|A_{(-)d} >$$

$$= \mathbf{g}^{(45)2}\{-\frac{1}{16}\left(\eta_{ab,cd}^{(45--)} + 4\eta_{ad,cb}^{(45--)}\right)(\mathbf{A}_{(-)ai}^{\dagger}\mathbf{A}_{(-)b}^{i}\mathbf{A}_{(-)c}^{jk\dagger}\mathbf{A}_{(-)djk} + \mathbf{A}_{(-)ai}^{\dagger}\mathbf{A}_{(-)b}^{i}\mathbf{A}_{(-)cj}^{\dagger}\mathbf{A}_{(-)d}^{j})$$

$$-\frac{1}{16}\left(-11\eta_{ab,cd}^{(45--)} + 4\eta_{ad,cb}^{(45--)}\right)\mathbf{A}_{(-)a}^{\dagger}\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^{ij\dagger}\mathbf{A}_{(-)dij}$$

$$-\frac{1}{2}\left(\eta_{ab,cd}^{(45--)} + \eta_{ad,cb}^{(45--)}\right)\mathbf{A}_{(-)ai}^{\dagger}\mathbf{A}_{(-)b}^{j}\mathbf{A}_{(-)c}^{ik\dagger}\mathbf{A}_{(-)dkj}$$

$$+\eta_{ab,cd}^{(45--)}[-\frac{1}{8}\epsilon_{ijklm}\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^{kl\dagger}\mathbf{A}_{(-)d}^{m} - \frac{1}{8}\epsilon^{ijklm}\mathbf{A}_{(-)ai}^{\dagger}\mathbf{A}_{(-)bjk}\mathbf{A}_{(-)c}^{\dagger}\mathbf{A}_{(-)dlm}$$

$$-\frac{1}{4}\mathbf{A}_{(-)a}^{ik\dagger}\mathbf{A}_{(-)bkj}\mathbf{A}_{(-)c}^{jl\dagger}\mathbf{A}_{(-)dli} + \frac{3}{64}\mathbf{A}_{(-)a}^{ij\dagger}\mathbf{A}_{(-)bij}\mathbf{A}_{(-)c}^{kl\dagger}\mathbf{A}_{(-)dkl}$$

$$+\frac{7}{8}\mathbf{A}_{(-)a}^{\dagger}\mathbf{A}_{(-)b}\mathbf{A}_{(-)ci}^{\dagger}\mathbf{A}_{(-)d}^{i} + \frac{15}{16}\mathbf{A}_{(-)a}^{\dagger}\mathbf{A}_{(-)b}\mathbf{A}_{(-)c}^{\dagger}\mathbf{A}_{(-)d}]\}(86)$$

$$-\frac{1}{16}\mathbf{g}^{(45)2}h_{ab}^{(45+)}h_{cd}^{(45-)} < A_{(+)a}|\Sigma_{\mu\nu}|A_{(+)b} >< A_{(-)c}|\Sigma_{\mu\nu}|A_{(-)d} >$$

$$= \mathbf{g}^{(45)2}\eta_{ab,cd}^{(45+-)}\{\frac{5}{8}\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}\mathbf{A}_{(-)c}^{\dagger}\mathbf{A}_{(-)d} - \frac{3}{32}\mathbf{A}_{(+)aij}^{\dagger}\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(-)c}^{kl\dagger}\mathbf{A}_{(-)dkl}$$

$$+\frac{1}{8}\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(-)cj}^{\dagger}\mathbf{A}_{(-)d}^{j} + \frac{1}{2}\mathbf{A}_{(+)a}^{j\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(-)cj}^{\dagger}\mathbf{A}_{(-)d}^{i}$$

$$+\frac{1}{2}\mathbf{A}_{(+)aik}^{\dagger}\mathbf{A}_{(+)b}^{kj}\mathbf{A}_{(-)c}^{il\dagger}\mathbf{A}_{(-)dlj} - \frac{3}{8}\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}\mathbf{A}_{(-)ci}^{\dagger}\mathbf{A}_{(-)d}^{i}$$

$$+\frac{13}{8}\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(-)c}^{\dagger}\mathbf{A}_{(-)d} + \frac{1}{16}\mathbf{A}_{(+)a}^{\dagger}\mathbf{A}_{(+)b}\mathbf{A}_{(-)c}^{ij\dagger}\mathbf{A}_{(-)dij}$$

$$+\frac{9}{16}\mathbf{A}_{(+)aij}^{\dagger}\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(-)c}^{\dagger}\mathbf{A}_{(-)d} - \frac{15}{16}\mathbf{A}_{(+)a}^{i\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(-)c}^{jk\dagger}\mathbf{A}_{(-)djk}$$

$$-\frac{1}{16}\mathbf{A}_{(+)aij}^{\dagger}\mathbf{A}_{(+)b}^{ij}\mathbf{A}_{(-)kc}^{\dagger}\mathbf{A}_{(-)d}^{k} + \frac{1}{2}\mathbf{A}_{(+)a}^{j\dagger}\mathbf{A}_{(+)bi}\mathbf{A}_{(-)c}^{ik\dagger}\mathbf{A}_{(-)dkj}$$

$$+\frac{1}{2}\mathbf{A}_{(+)aik}^{\dagger}\mathbf{A}_{(+)b}^{kj}\mathbf{A}_{(-)jc}^{\dagger}\mathbf{A}_{(-)d}^{i} - \frac{1}{4}\mathbf{A}_{(+)aij}^{\dagger}\mathbf{A}_{(+)b}\mathbf{A}_{(-)c}^{ij\dagger}\mathbf{A}_{(-)d}$$

27

$$-\frac{1}{4}\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{\dagger}_{(-)c}\mathbf{A}_{(-)dij} - \frac{1}{4}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{\dagger}_{(-)ci}\mathbf{A}_{(-)djk}$$

$$-\frac{1}{4}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bk}\mathbf{A}^{ij\dagger}_{(-)c}\mathbf{A}^{k}_{(-)d} - \frac{1}{2}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{\dagger}_{(-)cj}\mathbf{A}_{(-)dki}$$

$$-\frac{1}{2}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bk}\mathbf{A}^{ki\dagger}_{(-)c}\mathbf{A}^{j}_{(-)d} - \frac{1}{8}\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)b}\mathbf{A}^{\dagger}_{(-)ck}\mathbf{A}_{(-)dlm}$$

$$-\frac{1}{8}\epsilon_{ijklm}\mathbf{A}^{\dagger}_{(+)a}\mathbf{A}^{ij}_{(+)b}\mathbf{A}^{kl\dagger}_{(-)c}\mathbf{A}^{m}_{(-)d} - \frac{1}{8}\epsilon_{ijklm}\mathbf{A}^{i\dagger}_{(+)a}\mathbf{A}^{jk}_{(+)b}\mathbf{A}^{lm\dagger}_{(-)c}\mathbf{A}_{(-)d}$$

$$-\frac{1}{8}\epsilon^{ijklm}\mathbf{A}^{\dagger}_{(+)aij}\mathbf{A}_{(+)bk}\mathbf{A}^{\dagger}_{(-)c}\mathbf{A}_{(-)dlm}\}(87)$$

where η's are defined by

$$\eta^{(45++)}_{ab,cd} = h^{(45+)}_{ab} h^{(45+)}_{cd}; \qquad \eta^{(45--)}_{ab,cd} = h^{(45-)}_{ab} h^{(45-)}_{cd}; \qquad \eta^{(45+-)}_{ab,cd} = h^{(45+)}_{ab} h^{(45-)}_{cd} \qquad (88)$$

11 Appendix G

In this Appendix we discuss the coupling of the U(1) vector with matter without imposition of the constraint of the Wess-Zumino gauge. We consider the following Lagrangian which couples the vector multiplet with a scalar multiplet $\widehat{\Phi}$.

$$\mathsf{L}^{(U(1))} = \mathsf{L}^{(U(1)\ K.E.)}_V + \mathsf{L}^{(U(1)\ Mass)}_V + \mathsf{L}^{(U(1)\ Self-Interaction)}_V + \mathsf{L}^{(U(1)\ Interaction)}_{V+\Phi} + \mathsf{L}^{(U(1))}_\Phi$$

$$\mathsf{L}^{(U(1)\ K.E.)}_V = \frac{1}{4}\left[\widehat{\mathcal{W}}^{\tilde\alpha}\widehat{\mathcal{W}}_{\tilde\alpha}|_{\theta^2} + \widehat{\overline{\mathcal{W}}}_{\dot{\tilde\alpha}}\widehat{\overline{\mathcal{W}}}^{\dot{\tilde\alpha}}|_{\bar\theta^2}\right]$$

$$\mathsf{L}^{(U(1)\ Mass)}_V = m^2\widehat{\mathsf{V}}^2|_{\theta^2\bar\theta^2}$$

$$\mathsf{L}^{(U(1)\ Self-Interaction)}_V = \alpha_1\widehat{\mathsf{V}}^3|_{\theta^2\bar\theta^2} + \alpha_2\widehat{\mathsf{V}}^4|_{\theta^2\bar\theta^2}$$

$$\mathsf{L}^{(U(1)\ Interaction)}_{V+\Phi} = h\widehat{\Phi}^{\dagger}_a\widehat{\mathsf{V}}\widehat{\Phi}_a|_{\theta^2\bar\theta^2}$$

$$\mathsf{L}^{(U(1))}_\Phi = \widehat{\Phi}^{\dagger}_a\widehat{\Phi}_a|_{\theta^2\bar\theta^2} + \left[\mathsf{W}(\widehat{\Phi})|_{\theta^2} + h.c.\right](89)$$

where

$$\widehat{\mathcal{W}}_{\tilde\alpha} = -\frac{1}{4}\overline{\mathsf{D}}^2\mathsf{D}_{\tilde\alpha}\widehat{\mathsf{V}}; \quad \widehat{\overline{\mathcal{W}}}_{\dot{\tilde\alpha}} = -\frac{1}{4}\mathsf{D}^2\overline{\mathsf{D}}_{\dot{\tilde\alpha}}\widehat{\mathsf{V}} \qquad (90)$$

Finally, the superpotential $\mathsf{W}(\widehat{\Phi})$ of the theory is

$$\mathsf{W}(\widehat{\Phi}) = \mathcal{F}_a\widehat{\Phi}_a + \frac{1}{2}\mathcal{M}_{ab}\widehat{\Phi}_a\widehat{\Phi}_b + \frac{1}{3}\mathcal{G}_{abc}\widehat{\Phi}_a\widehat{\Phi}_b\widehat{\Phi}_c \qquad (91)$$

The couplings $\mathcal{M}_{ab}$ and $\mathcal{G}_{abc}$ are taken to be completely symmetric tensors. Expansion in component form gives

$$\mathsf{L}^{(U(1)\ K.E.)}_V = -\frac{1}{4}\mathcal{V}_{AB}\mathcal{V}^{AB} + \frac{1}{2}D^2 - i\lambda\sigma^A\partial_A\overline{\lambda} \qquad (92)$$

28

$$\mathsf{L}_V^{(U(1)\ Mass)} = m^2 CD + \frac{1}{2}m^2\left(M^2 + N^2\right) - m^2\left(\lambda\chi + \overline{\lambda}\overline{\chi}\right) - \frac{1}{2}m^2\partial^A C\partial_A C$$
$$-im^2\chi\sigma^A\partial_A\overline{\chi} - \frac{1}{2}m^2\mathcal{V}_A\mathcal{V}^A \quad (93)$$

$$\mathsf{L}_{V+\Phi}^{(U(1)\ Interaction)} = -\frac{h}{2\sqrt{2}}\left(\chi\sigma^A\overline{\psi}_a\right)\partial_A A_a + \frac{h}{\sqrt{2}}\left(\chi\sigma^A\partial\overline{\psi}_a\right)A_a - \frac{ih}{\sqrt{2}}\left(\overline{\lambda}\psi_a\right)A_a$$
$$-\frac{h}{2\sqrt{2}}\left(\psi_a\sigma^A\overline{\chi}\right)\partial_A A_a^\dagger - \frac{h}{\sqrt{2}}\left(\psi_a\sigma^A\partial\overline{\chi}\right)A_a^\dagger + \frac{ih}{\sqrt{2}}\left(\lambda\psi_a\right)A_a^\dagger$$
$$+\frac{ih}{2}\mathcal{V}_A\left[\left(\partial^A A_a\right)A_a^\dagger - \left(\partial^A A_a^\dagger\right)A_a\right] + \frac{h}{2}\mathcal{V}_A\left(\psi_a\sigma^A\overline{\psi}_a\right)$$
$$-hC\left(\partial_A A_a^\dagger\right)\left(\partial_A A_a\right) - ihC\left(\psi_a\sigma^A\partial_A\overline{\psi}_a\right) - \frac{h}{4}\partial^A\left(A_a A_a^\dagger\right)\partial_A C$$
$$+hCF_a F_a^\dagger - \frac{ih}{\sqrt{2}}\left(\chi\psi_a\right)F_a^\dagger + \frac{ih}{\sqrt{2}}\left(\overline{\chi}\overline{\psi}_a\right)F_a + \frac{h}{2}DA_a A_a^\dagger$$
$$+\frac{ih}{2}\left(M + iN\right)A_a F_a^\dagger - \frac{ih}{2}\left(M - iN\right)A_a^\dagger F_a \quad (94)$$

$$\mathsf{L}_V^{(U(1)\ Self-Interaction)} = -3\left(\frac{\alpha_1}{2}C + \alpha_2 C^2\right[\mathcal{V}_A\mathcal{V}^A + 2\left(\lambda\chi + \overline{\lambda}\overline{\chi}\right) + 2i\chi\sigma^A\partial_A\overline{\chi}$$
$$-\left(M^2 + N^2\right)] + 3\left(\frac{\alpha_1}{4} + \alpha_2 C\right)\left[i\left(M + iN\right)\left(\overline{\chi}\overline{\chi}\right) - i\left(M - iN\right)\left(\chi\chi\right)\right.$$
$$-2\left(\chi\sigma^A\overline{\chi}\right)\mathcal{V}_A] + \left(\frac{3\alpha_1}{4}C + \alpha_2 C^2\right)\left[2CD - \partial_A C\partial^A C\right] + \frac{3\alpha_2}{2}\left(\chi\chi\right)\left(\overline{\chi}\overline{\chi}\right) \quad (95)$$

$$\mathsf{L}_\Phi^{(U(1))} = -\partial_A A_a^\dagger\partial^A A_a - i\overline{\psi}_a\overline{\sigma}^A\partial_A\psi_a + F_a^\dagger F_a$$
$$-\left(\frac{1}{2}\mathcal{M}_{ab} + \mathcal{G}_{abc}A_c\right)\psi_a\psi_b - \left(\frac{1}{2}\mathcal{M}_{ab}^* + \mathcal{G}_{abc}^* A_c^\dagger\right)\overline{\psi}_a\overline{\psi}_b$$
$$+\left(\mathcal{F}_a + \mathcal{M}_{ab}A_b + \mathcal{G}_{abc}A_c A_c\right)F_a + \left(\mathcal{F}_a^* + \mathcal{M}_{ab}^* A_b^\dagger + \mathcal{G}_{abc}^* A_c^\dagger A_c^\dagger\right)F_a^\dagger \quad (96)$$

Evaluation of Eq.(89) using Eqs.(90-96) gives in the four-component notation

$$\mathsf{L}^{(U(1))} = -\frac{1}{4}\mathcal{V}_{AB}\mathcal{V}^{AB} - \frac{1}{2}m^2\mathcal{V}_A\mathcal{V}^A - \frac{1}{2}\partial^A B\partial_A B - i\overline{\Lambda}\gamma^A\partial_A\Lambda - m\overline{\Lambda}\Lambda$$
$$-\partial^A A_a^\dagger\partial_A A_a - \frac{h}{m}B\partial^A A_a^\dagger\partial_A A_a$$
$$-\frac{h}{4m}\partial^A\left(A_a A_a^\dagger\right)\partial_A B + \frac{ih}{2}\left(A_a^\dagger\partial^A A_a - A_a\partial^A A_a^\dagger\right)\mathcal{V}_A$$
$$+\frac{h}{2m\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\gamma^A\Lambda_L\right)\partial_A A_a + \left(\overline{\Lambda}_L\gamma^A\Psi_{aL}\right)\partial_A A_a^\dagger\right]$$
$$+\frac{h}{m\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\gamma^A\partial_A\Lambda_L\right)A_a - \left(\overline{\Lambda}_L\gamma^A\partial_A\Psi_{aL}\right)A_a^\dagger\right]$$
$$-\frac{ih}{\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\Lambda_R\right)A_a - \left(\overline{\Lambda}_R\Psi_{aL}\right)A_a^\dagger\right]$$

29

$$-\left[\left(\frac{1}{2}\mathcal{M}_{ab}+\mathcal{G}_{abc}A_d\right)\overline{\Psi}_{aR}\Psi_{bL}+\left(\frac{1}{2}\mathcal{M}_{ab}^*+\mathcal{G}_{abc}^*A_d^\dagger\right)\overline{\Psi}_{aL}\Psi_{bR}\right]$$

$$-i\left(1+\frac{h}{m}B\right)\overline{\Psi}_{aL}\gamma^A\mathcal{D}_A\Psi_{aL}$$

$$-\frac{1}{m^3}\left(\frac{3\alpha_1}{4}B+\frac{\alpha_2}{m}B^2\right)\partial_A B\partial^A B-\frac{3}{m}\left(\frac{\alpha_1}{2}B+\frac{\alpha_2}{m}B^2\right)\mathcal{V}_A\mathcal{V}^A$$

$$+\frac{3}{m^3}\left(\alpha_1 B+\frac{2\alpha_2}{m}B^2\right)\left(i\overline{\Lambda}_L\gamma^A\partial_A\Lambda_L-m\overline{\Lambda}\Lambda\right)$$

$$+\frac{3}{m^2}\left(\frac{\alpha_1}{2}+\frac{2\alpha_2}{m}B\right)\left(\overline{\Lambda}_L\gamma^A\Lambda_L\right)\mathcal{V}_A+\frac{3\alpha_2}{2m^4}\left(\overline{\Lambda}_R^c\Lambda_L\right)\left(\overline{\Lambda}_L\Lambda_R^c\right)+\mathsf{L}_{auxiliary}^{(U(1))}\qquad(97)$$

where we have defined

$$\Lambda=\begin{pmatrix}m\chi_{\tilde\alpha}\\\overline{\lambda}^{\dot{\tilde\alpha}}\end{pmatrix},\quad\Psi_a=\begin{pmatrix}\psi_{a\tilde\alpha}\\\overline{\psi}_a^{\dot{\tilde\alpha}}\end{pmatrix},\quad B=mC,\quad\Lambda^c=\mathcal{C}\overline{\Lambda}^T,\quad\mathcal{C}=\begin{pmatrix}i\sigma^2&0\\0&i\overline\sigma^2\end{pmatrix},$$

$$\overline{\Lambda}=\Lambda^\dagger\gamma^0,\quad\Lambda_{R,L}=\frac{1\pm\gamma_5}{2}\Lambda\quad,\mathcal{D}_A=\partial_A-\frac{ig}{2}\left(1+\frac{g}{m}B\right)^{-1}\mathcal{V}_A\,(98)$$

and

$$\mathsf{L}_{auxiliary}^{(U(1))}=\left(mB+\frac{3\alpha_1}{2m^2}B^2+\frac{2\alpha_2}{m^3}B^3+\frac{h}{2}A_a^\dagger A_a\right)D+\frac{1}{2}D^2$$

$$+\left(\frac{1}{2}m^2+\frac{3\alpha_1}{2m}B+\frac{3\alpha_2}{m^2}B^2\right)\left(M^2+N^2\right)$$

$$+i\left[\frac{h}{2}A_a F_a^\dagger+\frac{3}{m^2}\left(\frac{\alpha_1}{4}+\frac{\alpha_2}{m}B\right)\left(\overline{\Lambda}_L\Lambda_R^c\right)\right](M+iN)$$

$$-i\left[\frac{h}{2}A_a^\dagger F_a+\frac{3}{m^2}\left(\frac{\alpha_1}{4}+\frac{\alpha_2}{m}B\right)\left(\overline{\Lambda}_R^c\Lambda_L\right)\right](M-iN)$$

$$+\left[\mathcal{F}_a+\mathcal{M}_{ab}A_b+\mathcal{G}_{abc}A_b A_c+\frac{ih}{m\sqrt{2}}\left(\overline{\Lambda}_L\Psi_{aR}\right)\right]F_a$$

$$+\left[\mathcal{F}_a^*+\mathcal{M}_{ab}^*A_b^\dagger+\mathcal{G}_{abc}^*A_c^\dagger A_d^\dagger-\frac{ih}{m\sqrt{2}}\left(\overline{\Psi}_{aR}\Lambda_L\right)\right]F_a^\dagger+\left(\frac{h}{m}B+1\right)F_a^\dagger F_a\qquad(99)$$

We next eliminate the auxiliary fields M, N, and D through their field equations to get

$$\mathsf{L}_{auxiliary}^{(U(1))}=-\frac{1}{2}m^2 B^2-\frac{3\alpha_1}{2m}B^3-\left(\frac{9\alpha_1^2}{8m^4}+\frac{2\alpha_2}{m^2}\right)B^4-\frac{3\alpha_1\alpha_2}{m^5}B^5-\frac{2\alpha_2^2}{m^6}B^6$$

$$-\frac{h^2}{8}\left(A_a^\dagger A_a\right)\left(A_b^\dagger A_b\right)-\frac{mh}{2}B\left(A_a^\dagger A_a\right)-\frac{3h\alpha_1}{4m^2}B^2\left(A_a^\dagger A_a\right)-\frac{h\alpha_2}{m^3}B^3\left(A_a^\dagger A_a\right)$$

$$+\left[\mathcal{F}_a^*+\mathcal{M}_{ab}^*A_b^\dagger+\mathcal{G}_{abc}^*A_b^\dagger A_c^\dagger-\frac{ih}{m\sqrt{2}}\left(\overline{\Psi}_{aR}\Lambda_L\right)-\frac{hf_1(B)}{f_2(B)}\left(\overline{\Lambda}_R^c\Lambda_L\right)\right]F_a^\dagger$$

$$-\frac{2f_1^2(B)}{f_2(B)}\left(\overline{\Lambda}_R^c\Lambda_L\right)\left(\overline{\Lambda}_L\Lambda_R^c\right)\,(100)$$

30

where

$$f_1(B) = \frac{3}{m^2}\left(\frac{\alpha_1}{4} + \frac{\alpha_2}{m}B\right), \quad f_2(B) = m^2 + \frac{3\alpha_1}{m}B + \frac{6\alpha_2}{m^2}B^2 \tag{101}$$

and the auxiliary field F satisfies the field equation

$$F_b^\dagger\left[\delta_{ab}\left(1 + \frac{h}{m}B\right) - \frac{h^2}{2f_2(B)}A_a^\dagger A_b\right] = -\frac{ih}{m\sqrt{2}}\overline{\Lambda}_L\Psi_{aR} + \frac{hf_1(B)}{f_2(B)}\left(\overline{\Lambda}_L\Lambda_R^c\right)A_a^\dagger$$
$$-\mathcal{F}_a - \mathcal{M}_{ab}A_b - \mathcal{G}_{abc}A_bA_c \tag{102}$$

Inverting this last equation we obtain

$$F_a^\dagger = \left(1 + \frac{h}{m}B\right)^{-1}\left[\delta_{ab} + \frac{h^2 A_a^\dagger A_b}{2f_2^2(B)\left(1 + \frac{h}{m}B\right) - h^2 A_c^\dagger A_c}\right]$$
$$\times \left[-\frac{ih}{m\sqrt{2}}\overline{\Lambda}_L\Psi_{bR} + \frac{hf_1(B)}{f_2(B)}\left(\overline{\Lambda}_L\Lambda_R^c\right)A_b^\dagger - \mathcal{F}_b - \mathcal{M}_{bd}A_d - \mathcal{G}_{bde}A_dA_e\right] \tag{103}$$

For the case when self-interactions of the vector multiplet are absent (i.e., $\alpha_1 = \alpha_2 = 0$), we get

$$\mathsf{L}^{(U(1))} = -\frac{1}{4}\mathcal{V}_{AB}\mathcal{V}^{AB} - \frac{1}{2}m^2\mathcal{V}_A\mathcal{V}^A - \frac{1}{2}\partial^A B\partial_A B - i\overline{\Lambda}\gamma^A\partial_A\Lambda - m\overline{\Lambda}\Lambda$$

$$-\partial^A A_a^\dagger\partial_A A_a - \frac{h}{m}B\partial^A A_a^\dagger\partial_A A_a$$

$$-\frac{h}{4m}\partial^A\left(A_a A_a^\dagger\right)\partial_A B + \frac{ih}{2}\left(A_a^\dagger\partial^A A_a - A_a\partial^A A_a^\dagger\right)\mathcal{V}_A$$

$$+\frac{h}{2m\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\gamma^A\Lambda_L\right)\partial_A A_a + \left(\overline{\Lambda}_L\gamma^A\Psi_{aL}\right)\partial_A A_a^\dagger\right]$$

$$+\frac{h}{m\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\gamma^A\partial_A\Lambda_L\right)A_a - \left(\overline{\Lambda}_L\gamma^A\partial_A\Psi_{aL}\right)A_a^\dagger\right]$$

$$-\frac{ih}{\sqrt{2}}\left[\left(\overline{\Psi}_{aL}\Lambda_R\right)A_a - \left(\overline{\Lambda}_R\Psi_{aL}\right)A_a^\dagger\right]$$

$$-\left[\left(\frac{1}{2}\mathcal{M}_{ab} + \mathcal{G}_{abc}A_c\right)\overline{\Psi}_{aR}\Psi_{bL} + \left(\frac{1}{2}\mathcal{M}_{ab}^* + \mathcal{G}_{abc}^*A_c^\dagger\right)\overline{\Psi}_{aL}\Psi_{bR}\right]$$

$$-i\left(1 + \frac{h}{m}B\right)\overline{\Psi}_{aL}\gamma^A\mathcal{D}_A\Psi_{aL}$$

$$-\frac{1}{2}m^2 B^2 - \frac{h^2}{8}\left(A_a^\dagger A_a\right)\left(A_b^\dagger A_b\right) - \frac{h}{2}mB\left(A_a^\dagger A_a\right)$$

$$-\left[\frac{\delta_{ab}}{1 + \frac{h}{m}B} + \frac{h^2 A_a^\dagger A_b}{2m^2 - h^2\left(1 + \frac{h}{m}B\right)A_c^\dagger A_c}\right]\left[\mathcal{F}_a + \mathcal{M}_{ad}A_d + \mathcal{G}_{ade}A_dA_e + \frac{ih}{m\sqrt{2}}\overline{\Lambda}_L\Psi_{aR}\right]$$

$$\times\left[\mathcal{F}_b^* + \mathcal{M}_{bf}^*A_f^\dagger + \mathcal{G}_{bfg}^*A_f^\dagger A_g^\dagger - \frac{ih}{m\sqrt{2}}\left(\overline{\Psi}_{bR}\Lambda_L\right)\right] \tag{104}$$

As is evident the $U(1)$ invariant effective Lagrangian above is highly nonlinear with infinite order nonlinearities.

References

[1] J. Wess and B. Zumino, Nucl. Phys., B78(1974)1.

[2] H. Georgi, in Particles and Fields (edited by C.E. Carlson), A.I.P., 1975; H. Fritzch and P. Minkowski, Ann. Phys. **93**(1975)193.

[3] P. Nath and R. M. Syed, Phys. Lett. **B506**(2001)68.

[4] P. Nath and R. M. Syed, Nucl. Phys. B **618** (2001) 138.

[5] R.N. Mohapatra and B. Sakita, Phys. Rev.**D21**(1980)1062.

[6] F. Wilczek and A. Zee, Phys. Rev. **D25**(1982)553.

[7] For alternative approaches see, G.W. Anderson and T. Blazek, J. Math. Phys. **41** (2000)8170; hep-ph/0101349; C. S. Aulakh and A. Girdhar, arXiv:hep-ph/0204097.

[8] For application of large representations see, H. Georgi and D.V. Nanopoulos, Nucl. Phys. **B159**(1979)16; J.A. Harvey, P. Ramond, and D.B. Reiss, Phys. Lett. **B92**(1980)309; J.A. Harvey, D.B. Reiss and P. Ramond, Nucl. Phys. **B199**(1982)223; S.P. Martin, Phys. Rev. **46**(1992)2769; K.S. Babu and R.N. Mohapatra, Phys. Rev. Lett.**70**(1993)2845; C.S. Aulakh, A. Melfo, A. Rasin and G. Senjanovic, Phys. Lett. **B459**(1999)557; C. Aulakh, B. Bajc, A. Melfo, A. Rasin and G. Senjanovic, Nucl.Phys.**B597**(2001)89; M-C. Chen and K.T. Mahanthappa, Phys. Rev. **D62**(2000)113007; Phys. Rev. D **65** (2002) 053010; arXiv:hep-ph/0305088.

[9] A. Salam and J. Strathdee, Nucl. Phys. B **76** (1974) 477.

[10] P. Fayet, Nuovo Cim. A **31** (1976) 626; P. Fayet, LPTENS-84-17 *Presented at 3rd Trieste School on Supergravity and Supersymmetry, Trieste, Italy, Apr 4-14, 1984*